SOME REMARKS ON $\sigma(\phi(n))$

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It was conjectured in 1964 by A. Makowski & A. Schinzel [4] that, for every natural number n,

$$\frac{\sigma(\phi(n))}{n} \geq \frac{1}{2}. \tag{1}$$

They remarked also that even the weaker result

$$\inf \frac{\sigma(\phi(n))}{n} > 0 \tag{2}$$

is still unproved. Carl Pomerance [5] gave a proof of (2). Also S. W. Graham et al. [2] stated in the abstract, their result,

$$.576 < \liminf \frac{\sigma(\phi(P_m))}{P_m} \leq \limsup \frac{\sigma(\phi(P_m))}{P_m} \leq 1,$$

where P_m is the product of the first m primes.

Notation: We use p and q to denote exclusively primes, $m|n$ to denote m dividing n, and $m\nmid n$ to denote m not dividing n. We use $p^a\|n$ to mean $p^a|n$ and $p^{a+1}\nmid n$. Also n is k-full means $p|n$ impli es $p^k|n$

First, we observe that validity of (1) for all $n \geq 1$ implies

$$\frac{\sigma(\phi(n))}{n} \geq 1, \tag{3}$$

for odd n. This can be seen easily from the fact that when n is odd, $\phi(2n) = \phi(n)$. On the other hand, (3) implies (1) can be seen with the help of (4) below. It also implies that (1) is a strict inequality if $4|n$.

As in [5], we factor $\frac{\sigma(\phi(n))}{n}$ and obtain

$$\begin{aligned}\frac{\sigma(\phi(n))}{n} &= \frac{\sigma(\phi(n))}{\phi n}\frac{\phi(n)}{n} \\ &= \prod_{p^a\|\phi(n)}\left(1+\frac{1}{p}+\frac{1}{p^2}+\cdots+\frac{1}{p^a}\right)\prod_{p|n}\left(1-\frac{1}{p}\right) \qquad (4)\\ &= \prod_{p|n}\left(1-\frac{1}{p}\right)\prod_{p|\phi(n)}\left(1-\frac{1}{p}\right)^{-1}\prod_{p^a\|\phi(n)}\left(1-\frac{1}{p^{a+1}}\right) \qquad (5)\end{aligned}$$

and it follows that if n is k-full, $k \geq 2$, then

$$\frac{\sigma(\phi(n))}{n} \geq \frac{1}{\zeta(2)}\prod_{p|\phi(n);\, p\nmid n}\left(1-\frac{1}{p}\right)^{-1} \geq \frac{1}{\zeta(2)},$$

for any n. (Of course, for odd and k-full n or k-full n with particular prime factors, we get better bounds from here.) In Theorem 1 below, we improve this bound in the case of any k-full n, for $k \geq 3$.

We can see from (5) that the essential problem is to prove the inequality

$$\prod_{p|n}\left(1-\frac{1}{p}\right) \geq \prod_{q|\prod_{p|n}(p-1)}\left(1-\frac{1}{q}\right). \tag{6}$$

In fact, the conjecture for odd n (3), implies (6) for odd n. On the other hand, it is clear that, with the help of (5), (6) implies (1) with $1/(2\zeta(2))$ on the right side in place of $1/2$. Pomerance interprets (6) as follows: For odd $n \geq 1$, $\phi(n) \geq$ the geometric mean of n and $\phi(\phi(n))$.

We mention the following consequence of the conjecture. Call a set of primes $S = \{2 = q_1, q_2, \ldots, q_t\}$ self-filled if, for any prime p, $p|\prod_{r=1}^{t}(q_r - 1)$ implies $p \in S$. The sets $\{2\}, \{2, 3, 7\}$ are, for example, self-filled sets. Let S as above be a self-filled set. Let $T = \{p_1, p_2, \cdots\}$ be the set of primes of the form $p_r = q_1^{a_1}q_2^{a_2}\cdots q_t^{a_t}+1$ for $a_1 \geq 1$ and the other $a_r \geq 0$. Observe that, for $r \geq 2$, $q_r \in T$. Then the conjecture implies

$$\prod_{p\in T;\, p\notin S}\left(1-\frac{1}{p}\right) \geq \frac{1}{2}. \tag{7}$$

Indeed, assume the conjecture (3) holds for odd n. Let $n = \prod_{p\in T;\, p<x} p$. With the help of (5), we see that (3) implies (6) which, in turn, implies (7) since $q_r \in T$ for $r \geq 2$ and x is arbitrary. [Observe that, when x is large enough, the set of prime factors of $\prod_{p\in T;\, p<x}(p-1)$ is precisely S.] When $S = \{2\}$, the corresponding set $T = T_2$ is the set of Fermat primes for which (7) is valid. This is easily checked thus,

$$\prod_{p\in T_2;\, p\notin S}\left(1-\frac{1}{p}\right) \geq \lim_{t\to\infty}\prod_{r=0}^{t}\frac{2^{2^r}}{2^{2^r}+1} = \lim_{t\to\infty}\frac{1}{2}\left(1-\frac{1}{2^{2^{t+1}}}\right)^{-1} = \frac{1}{2},$$

which is (7). This, of course, verifies (6) also, when n is composed only of primes of the form 2^a+1 and, hence, implies (1) with $1/(2\zeta(2))$ on the right side instead of $1/2$ there for such n.

Theorem 1: Let $k \geq 2$. For k-full n, we have

$$\frac{\sigma(\phi(n))}{n} \geq \zeta^{-1}(k).$$

Theorem 2: We have, for infinitely many primes P,

$$\frac{\sigma(\phi(P))}{P} \geq (1+o(1))e^{\gamma}\log\log P \quad \text{as } P \to \infty.$$

Also, for all large n, we have

$$\frac{\sigma(\phi(n))}{n} \leq (1+o(1))e^{\gamma}\log\log n \quad \text{as } n \to \infty.$$

That is, the maximum order of $\frac{\sigma(\phi(n))}{n}$ is $e^{\gamma}\log\log n$.

Theorem 2 quantifies a result of Alaoglu & Erdös [1].

Proof of Theorem 1: Let $n = \prod_{r=1}^{k} p_r^{a_r}$. We note that $\prod_{r=1}^{k} p_r^{e_r}$ for $0 \le e_r \le a_r - 1$, $1 \le r \le k$, are different integers for different k-tuples $(e_1, e_2, \dots, e_k)$. Hence, the $a_1 a_2 \dots a_k$ integers $\prod_{r=1}^{k} p_r^{e_r}(p_r - 1)$ are all distinct. All these $a_1 a_2 \dots a_k$ integers are divisors of $\phi(n)$ as well. Therefore, we have $\sigma(\phi(n))$ at least as large as the sum of these divisors. That is,

$$\sigma(\phi(n)) \ge \prod_{r=1}^{k} \left((1 + p_r + \cdots + p_r^{a_r - 1})(p_r - 1) \right)$$
$$\ge n \prod_{r=1}^{k} \left(1 - \frac{1}{p_r^{a_r}} \right) \ge n / \zeta(k),$$

since $a_r \ge k$ for all r, and the proof of Theorem 1 is complete.

Proof of Theorem 2: Let $2 = p_1$, $3 = p_2, \dots$ be the sequence of primes. Let $Q_k = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$, where $a_r = 2\left[\frac{\log p_k}{\log p_r}\right] + 1$, so that

$$p_r^{a_r + 1} \ge p_k^2. \tag{8}$$

Let m be the least integer such that $P = P_k = Q_k m + 1$ is prime. We see that

$$Q_k \le \exp\left(\sum_{r=1}^{k} a_r \log p_r \right) \le p_k^{3k}$$

and hence, by the theorem on least primes in arithmetic progression (see, e.g., [3]), we obtain $P \le p_k^{30k}$ (we do not need the best exponent), and thus,

$$\log\log P \le (1 + o(1)) \log p_k \quad \text{as } k \to \infty. \tag{9}$$

Now, remembering that P is prime, we get, using (4), that

$$\frac{\sigma(\phi(P))}{P} = (1 - 1/P) \prod_{p^a \| \phi(P)} \left(1 + \frac{1}{p} + \cdots \frac{1}{p^a} \right)$$
$$\ge (1 + o(1)) \prod_{r=1}^{k} \left(1 + \frac{1}{p_r} + \cdots + \frac{1}{p_r^{a_r}} \right)$$
$$\ge (1 + o(1)) \prod_{r=1}^{k} \left(1 + \frac{1}{-p_r} \right)^{-1} \prod_{r=1}^{k} \left(1 - \frac{1}{p_r^{a_r + 1}} \right)$$
$$\ge (1 + o(1)) e^{\gamma} \log p_k,$$

using (8), and Mertens' theorem and the lower bound in Theorem 2 now follows from (9).

It follows from (5) that, for any n,

$$\frac{\sigma(\phi(n))}{n} \le \prod_{p | \phi(n)} \left(1 - \frac{1}{p} \right)^{-1} \le (1 + o(1)) e^{\gamma} \log\log \phi(n),$$

and the proof of Theorem 2 is complete.

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