FIBONACCI AUTOCORRELATION SEQUENCES

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1. INTRODUCTION AND GENERALITIES

For given nonnegative integers τ and n, let us define the elements $\Gamma_n(S_i, \tau)$ of the *Autocorrelation Sequences* of any sequence of numbers $\{S_i\}_0^\infty$.

Definition:

$$\Gamma_n(S_i, \tau) \stackrel{\text{def}}{=} \sum_{i=0}^{n} S_i S_{i+\tau} \quad (0 \le \tau \le n), \tag{1.1}$$

where the subscript $i+\tau$ must be considered as reduced modulo $n+1$.

Observe that Definition (1.1) differs from the definition of the *Cyclic Autocorrelation Function* for periodic sequences with period $n+1$, by the factor $1/(n+1)$ (e.g., see [2], p. 25).

It can readily be seen that Definition (1.1) can be written in the equivalent form

$$\Gamma_n(S_i, \tau) = \sum_{i=0}^{n-\tau} S_i S_{i+\tau} + \sum_{i=0}^{\tau-1} S_{i+n-\tau+1} S_i, \tag{1.1'}$$

where the second sum vanishes for $\tau = 0$. Moreover, we point out that the numbers $\Gamma_n(S_i, \tau)$ enjoy the following symmetry property

$$\Gamma_n(S_i, \tau) = \Gamma_n(S_i, n-\tau+1) \quad (0 < \tau \le n). \tag{1.2}$$

A numerical example will better clarify the above statements.

Example:

$$\begin{aligned}
\Gamma_5(S_i, 4) &= S_0S_4 + S_1S_5 + S_2S_0 + S_3S_1 + S_4S_2 + S_5S_3 && \text{[from (1.1)]} \\
&= (S_0S_4 + S_1S_5) + (S_0S_2 + S_1S_3 + S_2S_4 + S_3S_5) && \text{[from (1.1')]} \\
&= \Gamma_5(S_i, 2) = S_0S_2 + S_1S_3 + S_2S_4 + S_3S_5 + S_4S_0 + S_5S_1 && \text{[from (1.2)].}
\end{aligned}$$

For particular sequences S_i, a closed-form expression for $\Gamma_n(S_i, \tau)$ can readily be found. For example, if $S_i = i$ (the sequence of nonnegative integers), we have

$$\Gamma_n(i, \tau) = \{2n^3 - 3(\tau-1)(n^2-\tau) + n[3\tau(\tau-2)+1]\}/6. \tag{1.3}$$

Observe that, when $\tau = 0$, the identity (1.3) reduces to the well-known formula that gives the sum of the squares of the first n integers.

The aim of this paper is to establish closed-form expressions for the elements of the *Fibonacci Autocorrelation Sequences* $\{\Gamma_n(\tau)\}_\tau^\infty$ defined as

$$\Gamma_n(\tau) \stackrel{\text{def}}{=} \Gamma_n(F_i, \tau), \tag{1.4}$$

and to discover some properties of these integers (sections 2 and 3). In this paper, F_i and L_i will denote, as usual, the i^{th} Fibonacci number and Lucas number, respectively. The proofs of the obtained results are, in general, very lengthy and rather cumbersome and, in most cases, they must be split into four subcases according to the parity of τ and n. Sometimes the residue of n modulo 4 must also be taken into account. To save space, only one subcase for each proposition will be proved in full detail (section 4). The parity of $\Gamma_n(\tau)$ is also discussed (section 5), and a glimpse of possible future work concludes the paper (section 6).

The following Fibonacci identities will be used widely throughout the proofs:

$$\begin{cases} F_{h+k} + (-1)^k F_{h-k} = F_h L_k \\ F_{h+k} - (-1)^k F_{h-k} = L_h F_k \end{cases} \quad [3, \mathrm{I}_{21} - \mathrm{I}_{24}, \text{ p. } 59], \tag{1.5}$$

$$\begin{cases} L_{h+k} + (-1)^k L_{h-k} = L_h L_k \\ L_{h+k} - (-1)^k L_{h-k} = 5F_h F_k \end{cases} \quad [4, \text{ (17a) and (17b)}], \tag{1.6}$$

$$\sum_{j=1}^{k} F_{mj+n} = \frac{F_{m(k+1)+n} - (-1)^m F_{mk+n} - F_{m+n} + (-1)^m F_n}{L_m - (-1)^m - 1} \quad [1, \text{ (11)}], \tag{1.7}$$

$$\sum_{j=1}^{k} L_{mj+n} = \frac{L_{m(k+1)+n} - (-1)^m L_{mk+n} - L_{m+n} + (-1)^m L_n}{L_m - (-1)^m - 1} \quad [1, \text{ (15)}]. \tag{1.8}$$

2. CLOSED-FORM EXPRESSIONS FOR $\Gamma_n(\tau)$

In this section closed-form expressions for $\Gamma_n(\tau)$ are established and some particular cases are discussed. First of all, we show in Table 1 the integers $\Gamma_n(\tau)$ for the first few values of τ and n. The results presented in this section and in the rest of the paper can be readily checked against this table.

TABLE 1. The Numbers $\Gamma_n(\tau)$ for $0 \le \tau,\ n \le 10$

$\tau \rightarrow$

	0										
	1	0									
	2	1	1								
	6	3	4	3							
	15	9	8	8	9						
n	40	24	20	16	20	24					
$\downarrow$	104	64	47	37	37	47	64				
	273	168	117	84	78	84	117	168			
	714	441	293	202	165	165	202	293	441		
	1870	1155	748	495	374	330	374	495	748	1155	
	4895	3025	1924	1244	877	707	707	877	1244	1924	3025

By (1.1′), the numbers $\Gamma_n(\tau)$ can be expressed as

$$\Gamma_n(\tau) = \sum_{i=0}^{n-\tau} F_i F_{i+\tau} + \sum_{i=0}^{\tau-1} F_{i+n-\tau+1} F_i. \tag{2.1}$$

With the aid of the Binet form for F_i, (2.1) becomes

$$\begin{aligned}\Gamma_n(\tau) &= \frac{1}{5}\sum_{i=0}^{n-\tau}\left(\alpha^{2i+\tau} + \beta^{2i+\tau} - \alpha^i\beta^{i+\tau} - \beta^i\alpha^{i+\tau}\right)\\ &+ \frac{1}{5}\sum_{i=0}^{\tau-1}\left(\alpha^{2i+n-\tau+1} + \beta^{2i+n-\tau+1} - \alpha^i\beta^{i+n-\tau+1} - \beta^i\alpha^{i+n-\tau+1}\right),\end{aligned} \tag{2.2}$$

where $\alpha = 1-\beta = (1+\sqrt{5})/2$. By (2.2), using the Binet form for L_i, yields

$$\Gamma_n(\tau) = \frac{1}{5}\sum_{i=0}^{n-\tau} L_{2i+\tau} - \frac{1}{5}\sum_{i=0}^{n-\tau}(-1)^i L_\tau + \frac{1}{5}\sum_{i=0}^{\tau-1} L_{2i+n-\tau+1} - \frac{1}{5}\sum_{i=0}^{\tau-1}(-1)^i L_{n-\tau+1},$$

whence, by means of (1.8), we obtain

$$\Gamma_n(\tau) = \frac{1}{5}[L_{2n-\tau+1} - L_{\tau-1} + L_{n+\tau} - L_{n-\tau} - X(n,\tau)], \tag{2.3}$$

where

$$X(n,\tau) = \begin{cases} L_\tau & (n \text{ even}) \\ 0 & (n \text{ odd}) \end{cases} \quad (\tau \text{ even}), \tag{2.4}$$

and

$$X(n,\tau) = \begin{cases} L_{n-\tau+1} & (n \text{ even}) \\ L_\tau + L_{n-\tau+1} & (n \text{ odd}) \end{cases} \quad (\tau \text{ odd}). \tag{2.4′}$$

Now, by virtue of (2.3)-(2.4′), (1.5), and (1.6), after some manipulations, we get

$$\Gamma_n(\tau) = \begin{cases} F_{n+1}F_{n-\tau} + F_nF_\tau & (n \text{ even}) \quad (2.5) \\ F_n(F_{n-\tau+1} + F_\tau) & (n \text{ odd}) \quad (2.5′) \end{cases} \quad (\tau \text{ even}),$$

and

$$\Gamma_n(\tau) = \begin{cases} F_nF_{n-\tau+1} + F_{n+1}F_{\tau-1} & (n \text{ even}) \quad (2.5″) \\ F_{n+1}(F_{n-\tau} + F_{\tau-1}) & (n \text{ odd}) \quad (2.5‴) \end{cases} \quad (\tau \text{ odd}).$$

The proofs of (2.5)-(2.5‴) are similar; thus, for the sake of brevity, we give only the proof of (2.5).

Proof of (2.5): By (2.3) and (2.4), we can write

$$\begin{aligned}\Gamma_n(\tau) &= \frac{1}{5}(L_{2n-\tau+1} - L_{\tau+1} + L_{n+\tau} - L_{n-\tau})\\ &= \frac{1}{5}[L_{n+1+(n-\tau)} - L_{n+1-(n-\tau)} + L_{n+\tau} - L_{n-\tau}],\end{aligned}$$

whence, by (1.6), we get (2.5). □

The complete factorization of $\Gamma_n(\tau)$ in terms of Fibonacci and Lucas numbers can be obtained only for the cases (2.5′) and (2.5‴). We have

$$\Gamma_n(\tau) = \begin{cases} F_n F_{(n+1)/2-\tau} L_{(n+1)/2} & (n \equiv 1 \pmod 4) \\ F_n F_{(n+1)/2} L_{(n+1)/2-\tau} & (n \equiv 3 \pmod 4) \end{cases} \quad (\tau \text{ even}) \tag{2.6}$$

and

$$\Gamma_n(\tau) = \begin{cases} F_{n+1} F_{(n-1)/2} L_{(n+1)/2-\tau} & (n \equiv 1 \pmod 4) \\ F_{n+1} F_{(n+1)/2-\tau} L_{(n-1)/2} & (n \equiv 3 \pmod 4) \end{cases} \quad (\tau \text{ odd}). \tag{2.6′}$$

Proof of (2.6): Let us rewrite (2.5′) as

$$\Gamma_n(\tau) = F_n\left[F_{(n+1)/2+[(n+1)/2-\tau]} + F_{(n+1)/2-[(n+1)/2-\tau]}\right]. \tag{2.7}$$

Recalling that

$$\frac{n+1}{2} \text{ is } \begin{cases} \text{even if } n \equiv 3 \pmod 4 \\ \text{odd if } n \equiv 1 \pmod 4, \end{cases} \tag{2.8}$$

and taking into account that τ is even, use (2.7) and (1.5) to obtain (2.6). □

An analogous argument leads to the proof of (2.6′).

2.1 Particular Cases

By (2.5)-(2.5‴), simplified expressions of $\Gamma_n(\tau)$ can be obtained for some particular values of τ. In light of (1.2), we confine ourselves to considering values of τ less than or equal to $(n+1)/2$. The following results have been obtained.

$$\Gamma_n(0) = F_n F_{n+1} \quad (\text{cf. } [3, I_3]), \tag{2.9}$$

$$\Gamma_n(1) = \begin{cases} F_n^2 & (n \text{ even}) \\ F_n^2 - 1 & (n \text{ odd}) \quad (\text{by using the } \textit{Simson formula } [3, I_{13}]), \end{cases} \tag{2.10}$$

$$\Gamma_n(2) = \begin{cases} F_{n+1}F_{n-2} + F_n & (n \text{ even}) \\ F_n(F_{n-1}+1) & (n \text{ odd}), \end{cases} \tag{2.11}$$

$$\Gamma_n(3) = \begin{cases} F_n F_{n-2} + F_{n+1} & (n \text{ even}) \\ F_{n+1}(F_{n-3}+1) & (n \text{ odd}), \end{cases} \tag{2.12}$$

and

$$\Gamma_n\left(\left\lfloor \frac{n+1}{2} \right\rfloor\right) = \begin{cases} F_{n+2}F_{n/2} & (n \equiv 0 \pmod 4) & (2.13) \\ 2F_{n+1}F_{(n-1)/2} & (n \equiv 1 \pmod 4) & (2.13′) \\ F_n F_{n/2+1} + F_{n+1}F_{n/2-1} & (n \equiv 2 \pmod 4) & (2.13″) \\ 2F_n F_{(n+1)/2} & (n \equiv 3 \pmod 4), & (2.13‴) \end{cases}$$

where $\lfloor \cdot \rfloor$ denotes the greatest integer function. The algebraic manipulations necessary to obtain (2.9)-(2.13‴) from (2.5)-(2.5‴) are not difficult and are omitted for brevity. Observe that, by using the Binet form, the identity (2.13″) can be restated in the equivalent form

$$\Gamma_n(n/2) = (L_{3n/2+2} - 2L_{n/2+1})/5 \quad (n \equiv 2 \pmod 4), \tag{2.14}$$

and that identities (2.13′) and (2.13‴) can be obtained immediately by the upper identity in (2.6′) and by the lower identity in (2.6), respectively, taking (2.8) into account.

3. SOME IDENTITIES INVOLVING THE NUMBERS $\Gamma_n(\tau)$

In this section we present some identities involving the numbers $\Gamma_n(\tau)$. The proofs of these results will be partially given in the next section.

First, let us state the recurrence relations

$$\Gamma_{n+1}(\tau) = \Gamma_n(\tau) + \Gamma_{n-1}(\tau) + \frac{1}{5}(2L_{2n-\tau} + L_{\tau+(-1)^{n+\tau}}), \tag{3.1}$$

$$\Gamma_n(\tau+1) = \Gamma_n(\tau-1) - \Gamma_n(\tau) - F_\tau + \frac{1}{5}(2L_{n+\tau} + (-1)^\tau L_{n-\tau+1}), \tag{3.2}$$

and

$$\Gamma_n(\tau) = \begin{cases} \Gamma_{n-1}(\tau) + \Gamma_{n-1}(\tau-1) + F_{n-\tau} + F_{\tau-1}[1-(-1)^n]/2 & (\tau \text{ even}) \\ \Gamma_{n-1}(\tau) + \Gamma_{n-1}(\tau-1) + F_{\tau-1}[1+(-1)^n]/2 & (\tau \text{ odd}). \end{cases} \tag{3.3}$$

Remark 1: Observe that, since $\Gamma_n(\tau)$ has not been defined for $\tau > n$, the recurrence relations (3.1)-(3.3) make sense only for $0 \le \tau \le n-1$, due to the presence of the quantities $\Gamma_{n-1}(\tau)$ and $\Gamma_n(\tau+1)$.

Then, let us consider the sums along the rows, the columns, and the rising diagonals of the triangular array shown in Table 1. Define

$$R_n \overset{\text{def}}{=} \sum_{\tau=0}^{n} \Gamma_n(\tau), \tag{3.4}$$

$$C_k(\tau) \overset{\text{def}}{=} \sum_{n=\tau}^{k} \Gamma_n(\tau), \ (k \ge \tau), \tag{3.5}$$

and

$$D_n \overset{\text{def}}{=} \sum_{\tau=0}^{\lfloor n/2 \rfloor} \Gamma_{n-\tau}(\tau), \tag{3.6}$$

and state the following propositions.

Proposition 1: $R_n = (F_{n+2} - 1)^2$.

By Proposition 1 and [3, I_3 and I_1], it can readily be seen that the sum of all elements of Table 1 from the 0^{th} row to the k^{th} row (inclusive) is

$$\sum_{n=0}^{k} R_n = \sum_{\tau=0}^{k} C_k(\tau) = F_{k+2}F_{k+3} - 2F_{k+4} + k + 4. \tag{3.7}$$

Proposition 2: If τ is even,

$$C_k(\tau)=\begin{cases}\Gamma_k(\tau)+\Gamma_k(\tau+1)-F_\tau[F_{\tau+1}+(k-\tau)/2] & (k\text{ even})\\ \Gamma_{k+1}(\tau+1)-F_\tau[F_{\tau+1}+(k-\tau+1)/2] & (k\text{ odd}),\end{cases}$$

whereas, if τ is odd,

$$C_k(\tau)=\begin{cases}\Gamma_{k+1}(\tau)+F_{\tau-1}(F_{k+1}-F_{\tau+2})-F_kF_{k-\tau+1}-F_\tau(k-\tau+1)/2 & (k\text{ even})\\ \Gamma_k(\tau)+F_{\tau-1}(F_{k+2}-F_{\tau+2})+F_kF_{k-\tau}-F_\tau(k-\tau)/2 & (k\text{ odd}).\end{cases}$$

Remark 2: For the same reason as that mentioned in Remark 1, the expression of $C_k(\tau)$ stated in Proposition 2 does not apply when $k=\tau$ is even, due to the presence of the addend $\Gamma_k(\tau+1)$. Of course, in this case, we have $C_k(k)=\Gamma_k(k)$.

Proposition 3:

$$D_n=\begin{cases}\dfrac{1}{5}\left[\dfrac{L_{2n+3}+nL_n-5F_{(n+6)/2}}{2}-3(F_n-1)\right] & (n\text{ even})\\ \dfrac{1}{5}\left[\dfrac{L_{2n+3}+(n-1)L_n-5F_{(n+3)/2}-r+1}{2}-3F_n\right]+1 & (n\text{ odd}),\end{cases}$$

where r denotes the residue of n modulo 4.

Finally, the following sums are considered:

$$A_n\stackrel{\text{def}}{=}\sum_{\tau=0}^{n}(-1)^\tau\Gamma_n(\tau),\tag{3.8}$$

$$B_n\stackrel{\text{def}}{=}\sum_{\tau=0}^{n}\binom{n}{\tau}\Gamma_n(\tau).\tag{3.9}$$

Proposition 4: $A_n=\begin{cases}F_nF_{n+1} & (n\text{ even})\\ (F_{n-1}+1)^2 & (n\text{ odd}).\end{cases}$

Proposition 5: $B_n=\begin{cases}\dfrac{1}{5}\left(L_{3n+2}-\dfrac{5F_{2n+2}+L_{n+1}}{2}\right) & (n\text{ even})\\ \dfrac{1}{5}\left[L_{3n+2}-\dfrac{L_{n+1}(5F_{n+1}-1)}{2}\right] & (n\text{ odd}).\end{cases}$

4. PROOFS

As mentioned in the Introduction, to save space, the identities stated in section 3 will be proved only for one case of the parity of τ and n. The interested reader can complete the proofs as an exercise.

Proof of (3.1): (τ and n even). By (2.3)-(2.4′), we can write

$$\begin{aligned}\Gamma_n(\tau)+\Gamma_{n-1}(\tau) &= \frac{1}{5}(L_{2n-\tau+1}+L_{2n-\tau-1}-2L_{\tau-1}+L_{n+\tau+1}-L_{n-\tau+1}-L_\tau-0)\\ &= \frac{1}{5}(L_{2n-\tau+3}-2L_{2n-\tau}-2L_{\tau-1}+L_{n+\tau+1}-L_{n-\tau+1}-L_\tau)\\ &= \Gamma_{n+1}(\tau)-\frac{1}{5}(2L_{2n-\tau}+L_{\tau-1}+L_\tau)\\ &= \Gamma_{n+1}(\tau)-\frac{1}{5}(2L_{2n-\tau}+L_{\tau+1}),\end{aligned}$$

whence the recurrence (3.1). □

Proof of (3.2): (τ even and n odd). By (2.3)-(2.4′) and (1.6), we can write

$$\begin{aligned}\Gamma_n(\tau)+\Gamma_n(\tau+1) &= \frac{1}{5}(L_{2n-\tau+2}-L_{\tau+1}+L_{n+\tau+2}-L_{n-\tau+1}-0-L_{\tau+1}-L_{n-\tau})\\ &= \frac{1}{5}[L_{2n-\tau+2}-L_{n-\tau+1}+(L_{n+\tau+2}-2L_{\tau+1}-L_{n-\tau})+(L_{n+\tau-1}-L_{n-\tau+2}-L_\tau)\\ &\qquad -(L_{n+\tau-1}-L_{n-\tau+2}-L_\tau)]\\ &= \Gamma_n(\tau-1)+\frac{1}{5}(L_{n+\tau+2}-2L_{\tau+1}-L_{n-\tau}-L_{n+\tau-1}+L_{n-\tau+2}+L_\tau)\\ &= \Gamma_n(\tau-1)+\frac{1}{5}[L_{n+\tau+2}-L_{n-\tau}-L_{n+\tau-1}+L_{n-\tau+2}-(L_{\tau+1}+L_{\tau-1})]\\ &= \Gamma_n(\tau-1)-F_\tau+\frac{1}{5}(L_{n+\tau+2}-L_{n+\tau-1}+L_{n-\tau+1})\\ &= \Gamma_n(\tau-1)-F_\tau+\frac{1}{5}(2L_{n+\tau}+L_{n-\tau+1}),\end{aligned}$$

whence the recurrence (3.2). □

Proof of (3.3): (τ and n even). By (2.5′) and (2.5‴), the right-hand side of (3.3) can be rewritten as

$$\begin{aligned}F_{n-1}(F_{n-\tau}+F_\tau)+F_n(F_{n-\tau}+F_{\tau-2})+F_{n-\tau} &= F_{n+1}F_{n-\tau}+F_{n-1}F_\tau+F_nF_{\tau-2}+F_{n-\tau}\\ &= F_{n+1}F_{n-\tau}+(F_n-F_{n-2})F_\tau+F_nF_{\tau-2}+F_{n-\tau}\\ &= (F_{n+1}F_{n-\tau}+F_nF_\tau)-F_{n-2}F_\tau+F_nF_{\tau-2}+F_{n-\tau}\\ &= \Gamma_n(\tau)-F_{n-2}F_\tau+F_nF_{\tau-2}+F_{n-\tau}\ \text{[by (2.5)]}.\end{aligned}$$

Now, it is sufficient to prove that $F_{n-\tau}+F_nF_{\tau-2}-F_{n-2}F_\tau=0$, that is

$$F_nF_{\tau-2}-F_{n-2}F_\tau=-F_{n-\tau}. \tag{4.1}$$

To do this, consider the Fibonacci identity

$$F_kF_h-F_{k-a}F_{h+a}=(-1)^{h+1}F_{k-h-a}F_a, \tag{4.2}$$

which can readily be proved by using the Binet form, and put $k=n, h=\tau-2$ (even, by hypothesis), and $a=2$ in (4.2) to obtain (4.1). □

Proof of Proposition 1: By (2.3) and [3, I_2],

$$5R_n = L_{2n+4} + 3 - 2L_{n+3} - \sum_{\tau=0}^{n} X(n, \tau). \tag{4.3}$$

If n is even, then by (2.4), (2.4′), and (1.8),

$$\sum_{\tau=0}^{n} X(n, \tau) = 2L_{n+1}$$

and Proposition 1 holds by [3, I_{16}]. If n is odd, then by (2.4), (2.4′), and (1.8),

$$\sum_{\tau=0}^{n} X(n, \tau) = 2(L_{n+1} - 2)$$

and Proposition 1 holds by [3, I_{17}]. □

Proof of Proposition 2: (τ even and k odd). Put $n = j + \tau - 1$ in (3.5), thus getting

$$C_k(\tau) = \sum_{j=1}^{k-\tau+1} \Gamma_{j+\tau-1}(\tau) = \sum_{j=1}^{(k-\tau+1)/2} [\Gamma_{2j+\tau-2}(\tau) + \Gamma_{2j+\tau-1}(\tau)]. \tag{4.4}$$

By (4.4), (2.3), and (2.4), we obtain

$$C_k(\tau) = \frac{1}{5} \sum_{j=1}^{(k-\tau+1)/2} (L_{4j+\tau-3} + L_{4j+\tau-1} - 2L_{\tau-1} + L_{2j+2\tau-2} + L_{2j+2\tau-1} - L_{2j-2} - L_{2j-1} - L_\tau),$$

whence, by (1.6),

$$\begin{aligned} C_k(\tau) &= \frac{1}{5} \sum_{j=1}^{(k-\tau+1)/2} (5F_{4j+\tau-2} - 5F_\tau + L_{2j+2\tau} - L_{2j}) \\ &= \sum_{j=1}^{(k-\tau+1)/2} F_{4j+\tau-2} - \frac{k-\tau+1}{2} F_\tau + \frac{1}{5} \sum_{j=1}^{(k-\tau+1)/2} (L_{2j+2\tau} - L_{2j}). \end{aligned} \tag{4.5}$$

By (4.5), using (1.7) and (1.8) yields

$$\begin{aligned} C_k(\tau) = {} & \frac{1}{5}(F_{2k-\tau+4} - F_{2k-\tau} - F_{\tau+2} + F_{\tau-2}) - F_\tau(k-\tau+1)/2 \\ & + \frac{1}{5}(L_{k+\tau+3} - L_{k+\tau+1} - L_{2\tau+2} + L_{2\tau} - L_{k-\tau+3} + L_{k-\tau+1} + 1), \end{aligned}$$

whence, by (1.5),

$$C_k(\tau) = \frac{1}{5}(L_{2k-\tau+2} - L_\tau) - F_\tau(k-\tau+1)/2 + \frac{1}{5}(L_{k+\tau+2} - L_{2\tau+1} - L_{k-\tau+2} + 1)$$

and, by (1.6) (recalling that, since τ is even by hypothesis, $L_{-\tau} = L_\tau$),

$$\begin{aligned} C_k(\tau) &= F_{k+1}F_{k+1-\tau} - F_\tau(k-\tau+1)/2 + F_{k+2}F_\tau - F_\tau F_{\tau+1} \\ &= F_{k+1}F_{k+1-\tau} + F_{k+2}F_\tau - F_\tau[F_{\tau+1} + (k-\tau+1)/2]. \end{aligned} \tag{4.6}$$

By (4.6) and (2.5″), we obtain the desired result,

$$C_k(\tau)=\Gamma_{k+1}(\tau+1)-F_\tau[F_{\tau+1}+(k-\tau+1)/2]. \ \square$$

Proof of Proposition 3: $[n\equiv 1 \pmod 4]$. By (3.6), let us write

$$D_n=\sum_{\tau=0}^{(n-1)/4}\Gamma_{n-2\tau}(2\tau)+\sum_{\tau=1}^{(n-1)/4}\Gamma_{n-2\tau+1}(2\tau-1).$$

By (2.3)-(2.4′), and considering that $L_{-1}=-1$, the above expression becomes

$$\begin{aligned}D_n&=\frac{1}{5}\sum_{\tau=1}^{(n-1)/4}(L_{2n-6\tau+1}-L_{2\tau-1}+L_n-L_{n-4\tau})+\frac{1}{5}(L_{2n}+1)\\&\quad+\frac{1}{5}\sum_{\tau=1}^{(n-1)/4}(L_{2n-6\tau+4}-L_{2\tau-2}+L_n-L_{n-4\tau+2}-L_{n-4\tau+3})\\&=\frac{1}{5}\sum_{\tau=1}^{(n-1)/4}(L_{2n-6\tau+1}+L_{2n-6\tau+4}-L_{2\tau}+2L_n-L_{n-4\tau+4}-L_{n-4\tau})+\frac{1}{5}(L_{2n}+1),\end{aligned}$$

whence, by (1.6),

$$D_n=\frac{1}{5}\sum_{\tau=1}^{(n-1)/4}(2L_{2n-6\tau+3}-3L_{n-4\tau+2}-L_{2\tau}+2L_n)+\frac{1}{5}(L_{2n}+1). \tag{4.7}$$

By using (1.8), the identity (4.7) can be rewritten as

$$\begin{aligned}D_n=\frac{1}{5}\Bigg[&2\frac{L_{(n-3)/2}-L_{(n+9)/2}-L_{2n-3}+L_{2n+3}}{16}-3\frac{-1-4-L_{n-2}+L_{n+2}}{5}\\&-(L_{(n+3)/2}-L_{(n-1)/2}-1)+\frac{n-1}{2}L_n+L_{2n+1}+1\Bigg].\end{aligned} \tag{4.8}$$

Now, after some formal manipulations in the subscripts of the Lucas numbers in (4.8) (e.g., rewrite $L_{(n+9)/2}-L_{(n-3)/2}$ as $L_{[(n-3)/2+3]+3}-L_{[(n-3)/2+3]-3}$), use (1.6) once again to obtain

$$\begin{aligned}D_n&=\frac{1}{5}\left[\frac{L_{2n}-L_{(n+3)/2}}{2}-3(F_n-1)-L_{(n+1)/2}+\frac{n-1}{2}L_n+L_{2n+1}+2\right]\\&=\frac{1}{5}\left[\frac{L_{2n}+(n-1)L_n-5F_{(n+3)/2}}{2}-3(F_n-1)-L_{(n+1)/2}+2\right]\\&=\frac{1}{5}\left[\frac{2L_{2n+1}+L_{2n}+(n-1)L_n-5F_{(n+3)/2}}{2}-3F_n\right]+1\\&=\frac{1}{5}\left[\frac{L_{2n+3}+(n-1)L_n-5F_{(n+3)/2}}{2}-3F_n\right]+1. \ \square\end{aligned}$$

Proof of Proposition 4: If n is even, by (3.8), (1.2), and (2.9), we have $A_n=\Gamma_n(0)=F_nF_{n+1}$. If n is odd, by (3.8), (2.9), (2.5′), and (2.5‴), we can write

$$
\begin{aligned}
A_n &= \Gamma_n(0) + \sum_{\tau=1}^{(n-1)/2} \Gamma_n(2\tau) - \sum_{\tau=1}^{(n+1)/2} \Gamma_n(2\tau-1) \\
&= F_n F_{n+1} + F_n \sum_{\tau=1}^{(n-1)/2} (F_{n-2\tau+1} + F_{2\tau}) - F_{n+1} \sum_{\tau=1}^{(n+1)/2} (F_{n-2\tau+1} + F_{2\tau-2}),
\end{aligned}
$$

whence, by (1.7),

$$
\begin{aligned}
A_n &= F_n F_{n+1} + 2F_n(F_{n+1} - F_{n-1} - 1) - 2F_{n+1}(F_{n+1} - F_{n-1} - 1) \\
&= F_n F_{n+1} - 2F_{n-1}(F_n - 1) \\
&= F_n(F_{n+1} - 2F_{n-1}) + 2F_{n-1} \\
&= F_n F_{n-2} + 2F_{n-1}.
\end{aligned} \tag{4.9}
$$

By virtue of the identity [3, I_{19}], (4.9) becomes

$$A_n = F_{n-1}^2 + (-1)^{n-1} F_1 + 2F_{n-1} = F_{n-1}^2 + 1 + 2F_{n-1} = (F_{n-1}+1)^2. \ \square$$

The proof of Proposition 5 concludes this section. Here, we need the following four Lucas identities whose proofs can be obtained with the aid of the Binet form and the binomial formula:

$$\sum_{i=0}^{m} \binom{m}{i} L_{k+i} = L_{2m+k}, \tag{4.10}$$

$$\sum_{i=0}^{m} \binom{m}{i} L_{k-i} = L_{m+k}, \tag{4.11}$$

$$\sum_{i=0}^{m/2} \binom{m}{2i} L_{2i} = (L_{2m} + L_m)/2 \quad (m \text{ even}), \tag{4.12}$$

$$\sum_{i=0}^{\lfloor (m-1)/2 \rfloor} \binom{m}{2i+1} L_{m-2i} = [L_{2m+1} + (-1)^m L_{m-1}]/2. \tag{4.13}$$

Proof of Proposition 5: (n even). By (3.9), (2.3), (4.10), and (4.11), we readily obtain

$$B_n = \frac{1}{5}\left[L_{3n+2} - L_{2n+1} - \sum_{\tau=0}^{n} \binom{n}{\tau} X(n,\tau)\right]$$

and, by (2.4) and (2.4′),

$$B_n = \frac{1}{5}\left\{L_{3n+2} - L_{2n+1} - \left[\sum_{\tau=0}^{n/2} \binom{n}{2\tau} L_{2\tau} + \sum_{\tau=0}^{n/2-1} \binom{n}{2\tau+1} L_{n-2\tau}\right]\right\}. \tag{4.14}$$

Using (4.12) and (4.13), the equality (4.14) can be rewritten as

$$
\begin{aligned}
B_n &= \frac{1}{5}[L_{3n+2} - L_{2n+1} - (L_{2n} + L_n + L_{2n+1} + L_{n-1})/2] \\
&= \frac{1}{5}[L_{3n+2} - (2L_{2n+1} + L_{2n+2} + L_{n+1})/2] = \frac{1}{5}[L_{3n+2} - (L_{2n+1} + L_{2n+3} + L_{n+1})/2].
\end{aligned} \tag{4.15}
$$

Rewrite (4.15) as

$$B_n = \frac{1}{5}[L_{3n+2} - (L_{(2n+2)+1} + L_{(2n+2)-1} + L_{n+1})/2],$$

and use (1.6) to obtain the desired result,

$$B_n = \frac{1}{5}[L_{3n+2} - (5F_{2n+2} + L_{n+1})/2]. \quad \square$$

5. ON THE PARITY OF $\Gamma_n(\tau)$

The problem of establishing necessary and sufficient conditions for $\Gamma_n(\tau)$ to be divisible by a given integer k is believed to deserve a thorough investigation. Nevertheless, the general solution (if any) of this problem is beyond the scope of this paper. In this section we confine ourselves to solving the case $k=2$. The proofs of the results shown in the sequel are based on the well-known fact that

$$F_m \text{ is even if and only if } m \equiv 0 \pmod 3. \tag{5.1}$$

5.1 Results

The integer $\Gamma_n(\tau)$ is even if and only if

(i) n and τ even

$$n \equiv \begin{cases} 0 \\ 1 \\ 2 \end{cases} \pmod 3 \quad \text{and} \quad \tau \equiv \begin{cases} 0 \\ 2 \\ 0 \end{cases} \pmod 3,$$

(ii) n even and τ odd

$$n \equiv \begin{cases} 0 \\ 1 \\ 2 \end{cases} \pmod 3 \quad \text{and} \quad \tau \equiv \begin{cases} 1 \\ 0 \\ 0 \end{cases} \pmod 3,$$

(iii) n odd and τ even

$$n \equiv \begin{cases} 0 \\ 1 \\ 2 \end{cases} \pmod 3 \quad \text{and} \quad \tau \equiv \begin{cases} 0, 1, \text{ or } 2 \\ 1 \\ 0, 1, \text{ or } 2 \end{cases} \pmod 3,$$

(iv) n and τ odd

$$n \equiv \begin{cases} 0 \\ 1 \\ 2 \end{cases} \pmod 3 \quad \text{and} \quad \tau \equiv \begin{cases} 2 \\ 0, 1, \text{ or } 2 \\ 0, 1, \text{ or } 2 \end{cases} \pmod 3.$$

The above conditions on the parity of $\Gamma_n(\tau)$ are presented, in a more compact form, in Table 2, where h and k denote all nonnegative integers such that $0 \le \tau \le n$.

TABLE 2. Forms of n and τ for $\Gamma_n(\tau)$ To Be Even

n	τ		
$6h$	$6k$	or	$6k+1$
$6h+1$	$2k+1$	or	$6k+4$
$6h+2$	$6k$	or	$6k+3$
$6h+3$	$2k$	or	$6k+5$
$6h+4$	$6k+2$	or	$6k+3$
$6h+5$	k		

5.2 Proofs

The proofs of (iii) and (iv) are quite easy. The proofs of (i) and (ii) are similar, so we give only the latter in detail.

Proof of (ii): (n even and τ odd). By (2.5″) we see that $\Gamma_n(\tau)$ is even if and only if

$$\text{either } \begin{cases} A = F_n F_{n-\tau+1} & \text{is even} \\ B = F_{n+1} F_{\tau-1} & \text{is even} \end{cases} \text{(Case 1)}, \quad \text{or } \begin{cases} A & \text{is odd} \\ B & \text{is odd.} \end{cases} \text{(Case 2)}.$$

Case 1. A is even if and only if [see (5.1)]

$$\text{either } n \equiv 0 \pmod 3 \quad \text{or } n \equiv \tau - 1 \pmod 3,$$

whereas B is even if and only if

$$\text{either } n \equiv 2 \pmod 3 \quad \text{or } \tau \equiv 1 \pmod 3.$$

It follows that Case 1 occurs if and only if

$$n \equiv \begin{cases} 0 \\ 2 \end{cases} \pmod 3 \quad \text{and} \quad \tau \equiv \begin{cases} 1 \\ 0 \end{cases} \pmod 3. \tag{5.2}$$

Case 2. A is odd if and only if

$$n \not\equiv 0 \pmod 3 \quad \text{and} \quad n \not\equiv \tau - 1 \pmod 3,$$

whereas B is odd if and only if

$$n \not\equiv 2 \pmod 3 \quad \text{and} \quad \tau \not\equiv 1 \pmod 3.$$

It follows that Case 2 occurs if and only if

$$n \equiv 1 \pmod 3 \quad \text{and} \quad \tau \equiv 0 \pmod 3. \tag{5.3}$$

Combining (5.2) and (5.3) gives (ii). □

6. FURTHER WORK

Flowing from our development, there seem to be other possibilities for investigation. The main one among them consists of applying the operator Γ defined by (1.1) to other second-order recurring sequences, such as the Lucas sequence, the Pell sequence, and so on. As for the former, we obtained the identity

$$\Gamma_n(L_i, \tau) = 5\Gamma_n(\tau) + 2X(n, \tau) \quad \text{[cf. (2.4) and (2.4′)].} \tag{6.1}$$

On the other hand, we believe that our investigation of the numbers $\Gamma_n(\tau)$ deserves some further deepening. For example, on the bases of (2.5)-(2.5‴) and the identity $F_{-n} = (-1)^{n+1}F_n$, we can generalize these numbers to *any* integer value of the parameters τ and n (i.e., $\tau > n$ and n and/or $\tau < 0$). As a minor instance, it can be shown that

$$\Gamma_n(-n) = F_n(F_{n+1}L_n - F_n) \quad (n \text{ even}). \tag{6.2}$$

Moreover, the results presented in section 5 could be extended to the divisibility of $\Gamma_n(\tau)$ by $k > 2$. In particular, a study on the primality of these numbers should be undertaken. Early responses to this effort allow us to state the following necessary conditions for $\Gamma_n(\tau)$ to be a prime:

$$\begin{array}{l} n \text{ must be even} \\ \text{[with the unique exception } \Gamma_3(1) = \Gamma_3(3) = F_4(F_2 + F_0) = 3], \end{array} \tag{6.3}$$

$$\gcd(n - \tau, n) \le 2 \quad (\tau \text{ even}) \tag{6.4}$$

and

$$\begin{cases} \gcd(n, \tau - 1) \le 2 \\ \gcd(n - \tau + 1, n + 1) \le 2 \end{cases} (\tau \text{ odd}). \tag{6.5}$$

In passing, we observed that $\Gamma_n(0)$ is composite [except for $\Gamma_2(0) = F_2F_3 = 2$] and that $\Gamma_n(1)$ is composite as well, except for $\Gamma_2(1) = 1$.

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