ADVANCED PROBLEMS AND SOLUTIONS

Edited by Raymond E. Whitney

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-490 ***Proposed by A. Stuparu, Vâlcea, Romania***

Prove that the equation $S(x)=p$, where p is a given prime number, has just 2^{p-2} solutions, all of them in between p and $p!$. [$S(n)$ is the Smarandache Function: the smallest integer such that $S(n)!$ is divisible by n.]

H-491 ***Proposed by Paul S. Bruckman, Highwood, Illinois***

Prove the following identities:

$$F_{2n}=2\binom{2n}{n}^{-1}\sum_{k=0}^{n-1}\binom{n-\frac{1}{2}}{k}\binom{n-\frac{1}{2}}{n-1-k}5^k, \quad n=1,2,\ldots; \tag{a}$$

$$F_{2n+1}=\binom{2n}{n}^{-1}\sum_{k=0}^{n}\binom{n-\frac{1}{2}}{k}\binom{n+\frac{1}{2}}{n-k}5^k, \quad n=0,1,2,\ldots. \tag{b}$$

H-492 ***Proposed by H.-J. Seiffert, Berlin, Germany***

Define the Fibonacci polynomials by $F_0(x)=0$, $F_1(x)=1$, $F_n(x)=xF_{n-1}(x)+F_{n-2}(x)$, for $n\geq 2$. Show that, for all complex numbers x and y and all nonnegative integers n,

$$\sum_{k=0}^{[n/2]}\binom{n}{k}F_{n-2k}(x)F_{n-2k}(y)=z^{n-1}F_n(xy/z), \tag{1}$$

where $z=(x^2+y^2+4)^{1/2}$. [ ] denotes the greatest integer function.

As special cases of (1), obtain the following identities:

$$\sum_{k=0}^{[n/2]}\binom{n}{k}F_{n-2k}^2=(3^n-(-2)^n)/5, \tag{2}$$

$$\sum_{k=0}^{n}\binom{2n+1}{n-k}F_{2k+1}=5^n, \tag{3}$$

$$\sum_{k=0}^{n}\binom{2n}{n-k}F_{2k}F_{4k}=5^{n-1}(4^n-1) \tag{4}$$

$$\sum_{k=0}^{n}\binom{2n+1}{n-k}F_{2k+1}L_{4k+2}=5^n(2^{2n+1}+1), \tag{5}$$

$$\sum_{k=0}^{[n/2]}(-1)^k\binom{n}{k}F_{2n-4k}P_{n-2k}=F_n(6), \tag{6}$$

where $P_j=F_j(2)$ is the j^{th} Pell number,

$$\sum_{\substack{k=0\\(5,\,n-2k)=1}}^{[n/2]}(-1)^{[(n-2k+2)/5]}\binom{n}{k}=F_n. \tag{7}$$

The latter equation is the one given in H-444.

SOLUTIONS

Sum Problem

H-477 ***Proposed by Paul S. Bruckman, Edmonds, Washington (Vol. 31, no. 2, May 1993)***

Let

$$F_r(x)=z^r-\sum_{k=0}^{r-1}a_kz^{r-1-k}, \tag{1}$$

where $r\ge 1$, and the a_k's are integers.

Suppose F_r has distinct zeros θ_k, $k=1,2,\ldots,r$, and let

$$V_n=\sum_{k=1}^{r}\theta_k^n,\quad n=0,1,2,\ldots. \tag{2}$$

Prove that, for all primes p,

$$V_p\equiv a_0\ (\mathrm{mod}\ p). \tag{3}$$

Solution by H.-J. Seiffert, Berlin, Germany

From (1), it follows that

$$(-1)^ka_k=(-1)^ka_k(\theta_1,\ldots,\theta_r)=\sum_{1\le i_1<\cdots<i_{k+1}\le r}\theta_{i_1}\ldots\theta_{i_{k+1}}, \tag{4}$$

for $k=0,\ldots,r-1$, is the $(k+1)^{\text{th}}$ elementary symmetric polynomial. Let S_r denote the set of all permutations of $\{1,\ldots,r\}$. For the r-tuple $(j_1,\ldots,j_r)$, where $0\le j_1\le\cdots\le j_r<p$ and $j_1+\cdots+j_r=p$, we define an equivalence relation on S_r by $\pi\sim\sigma$ if and only if $(j_{\pi(1)},\ldots,j_{\pi(r)})=(j_{\sigma(1)},\ldots,j_{\sigma(r)})$. Let $A_1,\ldots,A_m$ denote the equivalence classes with respect to this equivalence relation. For each $n\in\{1,\ldots,m\}$, we choose a permutation $\pi_n\in A_n$. Then the polynomial

$$P_{j_1,\ldots,j_r}(\theta_1,\ldots,\theta_r)=\sum_{n=1}^{m}\theta_1^{j_{\pi_n(1)}}\cdots\theta_r^{j_{\pi_n(r)}}$$

is symmetric. By the fundamental theorem on symmetric polynomials (see A. I. Kostrikin, *Introduction to Algrbra* [Springer-Verlag, 1982], pp. 281-84), there exists a polynomial $Q_{j_1,\ldots,j_r}$ having integer coefficients such that [see (4)]

$$P_{j_1,\ldots,j_r}(\theta_1,\ldots,\theta_r)=Q_{j_1,\ldots,j_r}(a_0,\ldots,a_{r-1}). \tag{5}$$

The multinomial theorem gives

$$\left(\sum_{k=1}^{r}\theta_k\right)^p=\sum_{k=1}^{r}\theta_k^p+\sum_{\substack{0\le j_1,\ldots,j_r<p\\ j_1+\cdots+j_r=p}}\binom{p}{j_1,\ldots,j_r}\theta_1^{j_1}\cdots\theta_r^{j_r},$$

or, in view of (2) and after a little sorting,

$$a_0^p=V_p+\sum_{\substack{0\le j_1\le\cdots\le j_r<p\\ j_1+\cdots+j_r=p}}\binom{p}{j_1,\ldots,j_r}P_{j_1,\ldots,j_r}(\theta_1,\ldots,\theta_r). \tag{6}$$

Equations (5) and (6) show that V_p is indeed an integer. It is well known that

$$\binom{p}{j_1,\ldots,j_r}\equiv 0 \pmod p \tag{7}$$

for all primes p and r-tuples $(j_1,\ldots,j_r)$ with $0\le j_1,\ldots,j_r<p$ and $j_1+\cdots+j_r=p$. (5), (6), and (7) imply $V_p\equiv a_0^p \pmod p$. Using Fermat's little theorem, we obtain $V_p\equiv a_0 \pmod p$, the desired result. Finally, we note that the result remains true, if the zeros $\theta_1,\ldots,\theta_r$ of F_r are not distinct. In such cases, each zero of F_r must occur in the defini-tion of V_n respecting its multiplicity.

Comment on H-477: Using the result of H-477 (including my final remark), it is very easy to solve the following problem (O. Šuch, Problem 10268, *Amer. Math. Monthly* **99.10** [1992]:958).

> Define a sequence (V_n) by
>
> $$V_0=3, V_1=0, V_2=2, V_{n+3}=V_{n+1}+V_n, \text{ for all } n\ge 0.$$
>
> If p is a prime, show that $p|V_p$.

According to the result of H-477, we only have to show that

$$V_n=\theta_1^n+\theta_2^n+\theta_3^n,\ n\in N_0, \tag{8}$$

where

$$(z-\theta_1)(z-\theta_2)(z-\theta_3)=z^3-z-1. \tag{9}$$

To do so, it suffices to show that (8) holds for $n=0,1,2$. For $n=0$, (8) is true, since $V_0=3$. For $n=1$, it follows from (9) and $V_1=0$. From (9), we get $\theta_1\theta_2+\theta_2\theta_3+\theta_3\theta_1=-1$. Hence,

$$0=V_1^2=(\theta_1+\theta_2+\theta_3)^2=\theta_1^2+\theta_2^2+\theta_3^3+2(\theta_1\theta_2+\theta_2\theta_3+\theta_3\theta_1)=\theta_1^2+\theta_2^2+\theta_3^2-2$$

implies $V_2=2=\theta_1^2+\theta_2^2+\theta_3^2$.

Also solved by A. G. Dresel, F. J. Flanigan, L. Somer, L. Van Hamme, and the proposer.

String Along

H-478 ***Proposed by Gino Taddei, Rome, Italy***
(Vol. 31, no. 3, August 1993)

Consider a string constituted by h labeled cells $c_1, c_2, \ldots, c_h$. Fill these cells with the natural numbers 1, 2, ..., h according to the following rule: 1 in c_1, 2 in c_2, 3 in c_4, 4 in c_7, 5 in c_{11}, and so on. Obviously, whenever the subscript j of c_j exceeds h, it must be considered as reduced modulo h. In other words, the integer n $(1 \le n \le h)$ enters the cell $c_{j(n,h)}$, where

$$j(n,h) = \left\langle \frac{n^2-n+2}{2} \right\rangle_h,$$

and the symbol $\langle a \rangle_b$ denotes a if $a \le b$, and the remainder of a divided by b if $a > b$.

Determine the set of all values of h for which, at the end of the procedure, each cell has been entered by exactly one number.

Solution by Paul S. Bruckman, Highwood, Illinois

Let $U(h) = \bigcup_{n=1}^{h}\{j(n,h)\}$ and $V(h) = \{1, 2, \ldots, h\}$. We seek to characterize the set

$$S = \{h \in Z^+ : U(h) = V(h)\}.$$

Clearly, $1 \in S, 2 \in S$.

First, we show that, if $h \in S$, $h > 1$, then h must be even. Suppose $h > 1$ is odd. Clearly, $j(1,h) = 1$ for all h. Also, $j(h,h) = \langle h \cdot \frac{1}{2}(h-1)+1 \rangle_h = 1$, since $\frac{1}{2}(h-1)$ is an integer. Since $h > 1$, c_1 and c_h are distinct cells; however, they are both occupied by the number 1, which shows that $h \notin S$ if h is odd and $h > 1$.

Suppose $h \equiv 2^r \pmod{2^{2r+1}}$, $r = 0, 1, 2, \ldots$ Then

$$j(h \cdot 2^{-r}, h) = \left\langle \frac{h}{2^{r+1}}\left(\frac{h}{2^r}-1\right)+1 \right\rangle_h = \left\langle h \cdot \left(\frac{h-2^r}{2^{2r+1}}\right)+1 \right\rangle_h = 1.$$

Also, $j(1,h) = 1$. The only way for cells c_1 and $c_{h \cdot 2^{-r}}$ to be identical is for $h = 2^r$; otherwise, $h \notin S$. In other words, all elements of S must be powers of 2.

Define the ordered h-tuple $W(h) = (j(1,h), j(2,h), \ldots, j(h,h)) = (1, 2, 4, \ldots)$, which orders the elements of $U(h)$ according to the cell numbers. We first show that, for all h,

$$W(2h) \equiv (W(h), W^*(h)) \pmod{h} \text{ where } W^*(h) \text{ denotes the transpose of } W(h)\ [W(h) \text{ in reversed order}]. \tag{1}$$

Proof of (1): We first observe that, if $1 \le n \le h$,

$$j(n, 2h) \equiv j(n,h) \pmod{h}. \tag{2}$$

Also, $j(2h+1-n, 2h) = \langle \frac{1}{2}(2h+1-n)(2h-n)+1 \rangle_{2h} = \langle 2h^2 - 2nh + h + \frac{1}{2}(n^2-n)+1 \rangle_{2h} = \langle h + \frac{1}{2}(n^2-n)+1 \rangle_{2h} \equiv$ $\langle \frac{1}{2}(n^2-n)+1+h \rangle_{2h} \equiv \langle \frac{1}{2}(n^2-n)+1 \rangle_h \pmod{h}$, or

$$j(2h+1-n, 2h) \equiv j(n,h) \pmod{h}, \quad 1 \le n \le h. \tag{3}$$

We see that (1) is a consequence of (2) and (3).

Suppose now that $h = 2^r, r \geq 2$. Then $j(n+h, 2h) = \langle \frac{1}{2}(n+h)(n+h-1)+1\rangle_{2h} = \langle \frac{1}{2}(n^2-n) + \frac{1}{2}h(2n-1) + \frac{1}{2}h^2 + 1\rangle_{2h} = \langle \frac{1}{2}(n^2-n) + 1 + \frac{1}{2}(h^2-h) + nh\rangle_{2h}$. If n is even, then $\frac{1}{2}(h^2-h)+nh \equiv \frac{1}{2}(h^2-h) \equiv 2^{r-1}(2^r-1) \pmod{2^{r+1}} \equiv -2^{r-1} \equiv -\frac{1}{2}h \pmod{2h}$; if n is odd, then $\frac{1}{2}(h^2-h)+nh \equiv \frac{1}{2}(h^2+h) \equiv 2^{r-1}(2^r+1) \pmod{2^{r+1}} \equiv 2^{r-1} \equiv \frac{1}{2}h \pmod{2h}$. In either case, we see that, if $h = 2^r$, $r \geq 2$, then $j(n, 2h) = j(n+h, 2h) + \frac{1}{2}h(-1)^n$, so

$$j(n, 2h) \neq j(n+h, 2h),\ 1 \leq n \leq h. \tag{4}$$

We may now complete the proof of the desired result, namely,

$$S = \{1, 2, 2^2, 2^3, \ldots\} = \text{the set of all nonnegative powers of 2.} \tag{5}$$

Our proof is by induction (on r). We suppose $h = 2^r, r \geq 0$, and $h \in S$. (Indeed, we already know that $1 \in S, 2 \in S$). Then the elements of $W(h)$ are distinct (mod h) and, *a fortiori*, (mod $2h$). Also, (1) holds. Therefore, the first (and also the last) h elements of $W(2h)$ are distinct. Moreover, it follows from (4) that the elements of the first half of $W(2h)$ are distinct from the elements of the second half of $W(2h)$. We conclude that $2h \in S$ as a consequence of $h \in S$. Since $W(4) = \{1, 2, 4, 3\}$, thus $4 \in S$. Then, by induction, (5) is established.

Also solved by P. G. Anderson, M. Barile, P. Filipponi, J. Hendel, N. Jensen, and A. N. 't Woord.

Close Ranks

H-479 ***Proposed by Richard André-Jeannin, Longwy, France (Vol. 31, no. 3, August 1993)***

Let $\{V_n\}$ be the sequence defined by $V_0 = 2$, $V_1 = P$, and $V_n = PV_{n-1} = QV_{n-2}$ for $n \geq 2$, where P and Q are real or complex parameters. Find a closed form for the sum

$$\sum_{k=1}^{n} \binom{2n-k-1}{n-1} P^k Q^{n-k} V_k.$$

Solution by Paul S. Bruckman, Everett, Washington

Let

$$S_n = \sum_{k=1}^{n} \binom{2n-k-1}{n-1} P^k Q^{n-k} V_k,\ n = 1, 2, \ldots. \tag{1}$$

Replacing k by $n-k$ yields

$$S_n = \sum_{k=0}^{n-1} \binom{n-1+k}{n-1} P^{n-k} Q^k V_{n-k}. \tag{2}$$

We seek to prove the following:

$$S_n = P^{2n},\ n = 1, 2, \ldots. \tag{3}$$

Toward this end, let

$$D_n = S_{n+1} - P^2 S_n,\ n = 1, 2, \ldots. \tag{4}$$

We may proceed to evaluate D_n in a straightforward manner, though not without some useful "tricks." Thus:

$$\begin{aligned}
D_n &= \sum_{k=0}^{n}\binom{n+k}{n}P^{n+1-k}Q^kV_{n+1-k} - \sum_{k=0}^{n-1}\binom{n-1+k}{n-1}P^{n+2-k}Q^kV_{n-k} \\
&= \sum_{k=1}^{n+1}\binom{n+k-1}{n}P^{n+2-k}Q^{k-1}V_{n+2-k} - \sum_{k=0}^{n-1}\binom{n+k-1}{n-1}P^{n+2-k}Q^kV_{n-k} \\
&= \binom{2n}{n}PQ^nV_1 + \binom{2n-1}{n}P^2Q^{n-1}V_2 - P^{n+2}V_n \\
&\qquad + \sum_{k=1}^{n-1}P^{n+2-k}Q^{k-1}\left[\binom{n+k-1}{n}V_{n+2-k} - \binom{n+k-1}{n-1}QV_{n-k}\right] \\
&= 2\binom{2n-1}{n}P^2Q^n + \binom{2n-1}{n}P^2Q^{n-1}(P^2-2Q) - P^{n+2}V_n \\
&\qquad + \sum_{k=1}^{n-1}P^{n+2-k}Q^{k-1}\left[\binom{n+k-1}{n}(PV_{n+1-k}-QV_{n-k}) - \binom{n+k-1}{n-1}QV_{n-k}\right] \\
&= \binom{2n-1}{n}P^4Q^{n-1} - P^{n+2}V_n + \sum_{k=0}^{n-2}\binom{n+k}{n}P^{n+2-k}Q^kV_{n-k} \\
&\qquad - \sum_{k=1}^{n-1}\left[\binom{n+k-1}{n} + \binom{n+k-1}{n-1}\right]P^{n+2-k}Q^kV_{n-k} \\
&= \binom{2n-1}{n}P^4Q^{n-1} - P^{n+2}V_n + \sum_{k=0}^{n-2}\binom{n+k}{n}P^{n+2-k}Q^kV_{n-k} - \sum_{k=1}^{n-1}\binom{n+k}{n}P^{n+2-k}Q^kV_{n-k} \\
&= \sum_{k=0}^{n-1}\binom{n+k}{n}P^{n+2-k}Q^kV_{n-k} - \sum_{k=0}^{n-1}\binom{n+k}{n}P^{n+2-k}Q^kV_{n-k} = 0.
\end{aligned}$$

We have tacitly assumed that $n \ge 2$ in the above development; it is a trivial exercise to verify that $S_1 = P^2$, $S_2 = P^4$. Therefore, by an easy induction, since $D_n = 0$ for all $n \ge 1$, (3) is established.

Also solved by P. Filipponi, N. Jensen, H.-J. Seiffert, A. Shannon, and the proposer.

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