FIBONACCI, LUCAS AND CENTRAL FACTORIAL NUMBERS, AND π

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In [1], a solution of Problem B-705, the evaluation of the series $\sum_{n=1}^{\infty} a_{2n}/(n^2\binom{2n}{n})$ for the Fibonacci numbers $a_n = F_n$ and the Lucas numbers $a_n = L_n$, proposed by H.-J. Seiffert, is given. The proof is essentially based on the power series expansion of $(\arcsin x)^2$. The same method yields, in the case $a_n = 1$, the Catalan-Apéry representation $\pi^2 = 18\sum_{k=1}^{\infty} k^{-2}/\binom{2k}{k}$ (see [3]).

Now it is possible to deduce a more general formula by using the Taylor series expansion ([2])

$$\left(2\arcsin\frac{x}{2}\right)^m = m!\sum_{k=m}^{\infty}\frac{|t(k,m)|}{k!}x^k, \quad |x|\le 2,\ m\in\mathbf{N}_0. \tag{1}$$

Here $t(k,m)$ denote the central factorial numbers of the first kind, which are defined by $(x^{[0]} := 1)$ (see [2], [4])

$$x^{[k]} := x\prod_{j=1}^{k-1}\left(x-\frac{k}{2}+j\right) = \sum_{m=0}^{k} t(k,m)x^m, \quad x\in\mathbf{R}.$$

Observing that $\arcsin(\alpha/2) = 3\pi/10$ and $\arcsin(\beta/2) = -\pi/10$, as well as Binet's formula $F_n = (\alpha^n - \beta^n)/\sqrt{5}$ and $L_n = \alpha^n + \beta^n$ $[\alpha = (1+\sqrt{5})/2,\ \beta = (1-\sqrt{5})/2]$, and noting the representation (1), one can readily deduce

Theorem 1: For $m\in\mathbf{N}$, there hold

$$\begin{aligned}\pi^m &= \frac{5^m m!\sqrt{5}}{3^m+(-1)^{m+1}}\sum_{k=m}^{\infty}\frac{F_k|t(k,m)|}{k!},\\ \pi^m &= \frac{5^m m!}{3^m+(-1)^m}\sum_{k=m}^{\infty}\frac{L_k|t(k,m)|}{k!}.\end{aligned} \tag{2}$$

The particular case $m = 1$ yields

$$\pi = \frac{5}{4}\sqrt{5}\sum_{k=0}^{\infty}\frac{F_{2k+1}}{2k+1}\frac{1}{16^k}\binom{2k}{k} = \frac{5}{2}\sum_{k=0}^{\infty}\frac{L_{2k+1}}{2k+1}\frac{1}{16^k}\binom{2k}{k}.$$

For $m = 2$, one obtains the solution of Problem B-705, and the case $m = 3$ in (2) gives, by formula (xxii) of [2],

$$\pi^3 = \frac{750}{7}\sqrt{5}\sum_{k=1}^{\infty}\frac{F_{2k+1}}{2k+1}\frac{1}{16^k}\binom{2k}{k}\sum_{j=1}^{k}\frac{1}{(2j-1)^2}.$$

Observe that, for large k, $\binom{2k}{k}/(16^k(2k+1)) \sim 4^{-k}k^{-3/2}$ (see [3]).

Vice versa, the Fibonacci and Lucas numbers can be expressed in terms of the central factorial numbers and π as follows.

Theorem 2: For $n \in \mathbf{N}$, there hold

$$F_n = \frac{(-1)^n n!}{\sqrt{5}} \sum_{k=n}^{\infty} i^{k+n} \frac{T(k,n)}{k!} \frac{\pi^k}{5^k}(3^k + (-1)^{k+1}),$$

$$L_n = (-1)^n n! \sum_{k=n}^{\infty} i^{k+n} \frac{T(k,n)}{k!} \frac{\pi^k}{5^k}(3^k + (-1)^k).$$

Here $T(k,n)$ are now the central factorial numbers of the second kind, given by

$$T(k,n) = \frac{1}{n!} \sum_{j=0}^{n} (-1)^j \binom{n}{j} \left(\frac{n}{2} - j\right)^k, \quad n, k \in \mathbf{N}_0.$$

The central factorial numbers of the first and second kind are connected by the orthogonality formula $\sum_{k=0}^{N} t(n,k) T(k,m) = \delta_{n,m}$, $N := \max(n,m)$ (see [2]; [4], p. 213).

To prove Theorem 2, one inserts the values $x = 3\pi/5$ and $x = -\pi/5$ into the expansion (see [2])

$$\left(2 \sin \frac{x}{2}\right)^n = (-1)^n n! \sum_{k=n}^{\infty} i^{k+n} \frac{T(k,n)}{k!} x^k, \quad x \in \mathbf{R},\ n \in \mathbf{N}_0,$$

and again uses the formula of Binet and $L_n = \alpha^n + \beta^n$, respectively.

REFERENCES

1. "Solution to Problem B-705: An Application of a Series Expansion for $(\arcsin x)^2$." (Proposed by H.-J. Seiffert.) *The Fibonacci Quarterly* **31** (1993):85-86.
2. P. L. Butzer, M. Schmidt, E. L. Stark, & L. Vogt. "Central Factorial Numbers: Their Main Properties and Some Applications." *Numer. Funct. Anal. Optim.* **10** (1989):419-88.
3. P. L. Butzer & M. Hauss. "Integral and Rapidly Converging Series Representations of the Dirichlet L-Functions $L_1(s)$ and $L_{-4}(s)$. *Atti Sem. Mat. Fis. Univ. Modena* **40** (1992):329-59.
4. J. Riordan. *An Introduction to Combinatorial Analysis.* New York: Wiley, 1958.

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