A NOTE ON A GEOMETRICAL PROPERTY OF FIBONACCI NUMBERS

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(*Submitted March 1993*)

INTRODUCTION

In [2] the authors, amid a more extensive analysis, prove an interesting geometrical property of Fibonacci numbers. They adopt the unusual convention (see [1] for the usual convention) that the Fibonacci sequence is given by

$$f_0 = f_1 = 1, \quad f_{n+2} = f_{n+1} + f_n, \quad n \geq 0, \tag{1}$$

Let F_n be the point (f_{n-1}, f_n) in the coordinate plane; let $X_n = (f_{n-1}, 0)$, $Y_n = (0, f_n)$; and let p_n be the broken line from O to F_n consisting of the straight line segments $OF_1, F_1F_2, \ldots, F_{n-1}F_n$. Then it is proved in [2] that p_n separates the rectangle $OX_nF_nY_n$ into two regions of equal area, provided that n is odd. Our main object in this note is to give an elementary geometrical proof of their quoted result, and then to give an elementary algebraic proof of a generalized version of this result.

PROOF WITHOUT WORDS

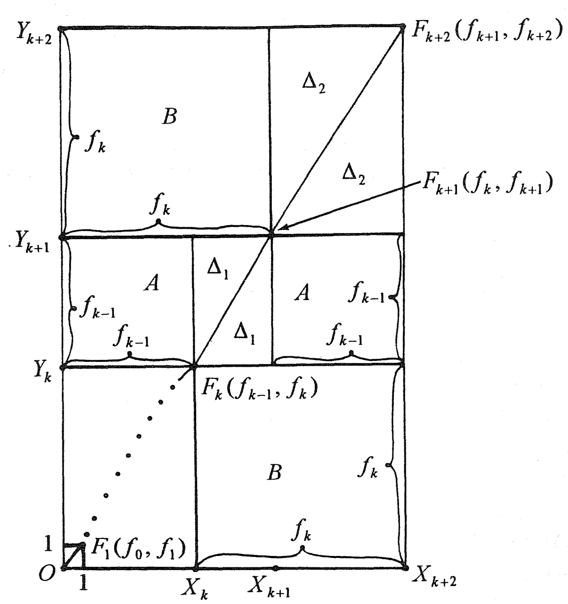

FIGURE 1

Figure 1 shows a path that begins at the origin and then progresses through the points $F_k(f_{k-1}, f_k)$, where the f_k are defined as in (1) above. We quote the first result of [2]:

. . .for each $n \geq 1$, the polygonal path $F_0F_1F_2 \cdots F_{2n+1}$ splits the rectangle $F_0X_{2n+1}F_{2n+1}Y_{2n+1}$ into two regions of equal area. (Note that, in [2], the origin is referred to as F_a.)

Inspection of Figure 1 (where congruent regions are labeled with the same symbol) reveals that the above result may be seen to be true by simply looking at the geometry of the suitably subdivided rectangle which evolves as a polygonal path passes from F_k through F_{k+1} to F_{k+2}. For Figure 1 clearly shows that, for *all* $k \geq 1$,

$$\text{area } Y_kF_kF_{k+1}F_{k+2}Y_{k+2} = \text{area } X_kF_kF_{k+1}F_{k+2}X_{k+2},$$

and hence it follows that, since the polygonal path from F_0 to F_1 obviously splits the rectangle $F_0X_1F_1Y_1$ into two regions of equal area, then the polygonal path from F_0 to F_{2k+1} splits the rectangle $F_0X_{2k+1}F_{2k+1}Y_{2k+1}$ into two regions of equal area. Notice that Figure 1 also tells us that the first line segment could have gone straight from F_0 to F_j, $j \geq 1$, and then the polygonal path from F_0 to F_{2k+j} would split the rectangle $F_0X_{2k+j}F_{2k+j}Y_{2k+j}$ into two regions of equal area. Furthermore, since the calculation of the lengths of the sides of the squares in Figure 1 depends effectively only on the recurrence relation in (1), and not on the initial values, any sequence of positive numbers (the Lucas sequence, for example) satisfying (1) will produce a similar result.

THE THEOREM

We consider *any* sequence $\{u_n\}$ of nonnegative numbers satisfying the recurrence relation $u_{n+2} = u_{n+1} + u_n$; notice that, in particular, we might consider the Fibonacci sequence or the Lucas sequence starting at *any* place along the sequence. We proceed exactly as in the Introduction, replacing f_n by u_n, so that $U_n = (u_{n-1}, u_n)$, $X_n(u_{n-1}, 0)$, $Y = (0, u_n)$, and the broken line $p_n = OU_1U_2 \ldots, U_n$ separates the rectangle $OX_nU_nY_n$ into two regions.

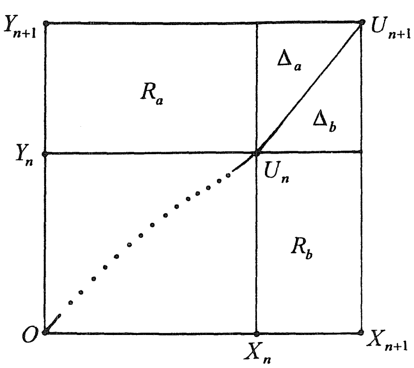

FIGURE 2

Theorem: The dotted line p_n separates the rectangle $OX_nU_nY_n$ into two regions of equal area, provided that n is odd.

We need the following simple lemma.

Lemma: $u_n^2 - u_{n+1}u_{n-1} = -(u_{n-1}^2 - u_nu_{n-2})$.

Proof of Lemma: $u_n^2 - u_{n+1}u_{n-1} = (u_n^2 - u_nu_{n-1}) - (u_{n+1}u_{n-1} - u_nu_{n-1}) = u_nu_{n-2} - u_{n-1}^2$.

Proof of Theorem: We argue by induction on n, the case $n=1$ being trivial. Consider the piece added on in passing from the rectangle $OX_nU_nY_n$ to the rectangle $OX_{n+1}U_{n+1}Y_{n+1}$. This may be subdivided, as in Figure 2, into a triangle $\triangle_a$ and a rectangle R_a above p_{n+1}, and a triangle $\triangle_b$ and a rectangle R_b below p_{n+1}. Obviously,

$$\begin{cases} \text{area } \triangle_a = \text{area } \triangle_b, \\ \text{area } R_a = u_{n-1}(u_{n+1}-u_n) = u_{n-1}^2, \\ \text{area } R_b = u_n(u_n - u_{n-1}) = u_n u_{n-2}. \end{cases} \tag{2}$$

Let A_n be the area of the region above p_n, and B_n the area of the region below p_n in the n^{th}-stage rectangle. We have proved that

$$A_{n+1} - B_{n+1} = A_n - B_n + D_n, \text{ where } D_n = u_{n-1}^2 - u_n u_{n-2}. \tag{3}$$

Now our Lemma asserts that

$$D_{n+1} = -D_n. \tag{4}$$

Thus, by (3) and (4),

$$A_{n+2} - B_{n+2} = A_n - B_n. \tag{5}$$

The equality (5) provides the inductive step to complete the proof.

REMARKS

(i) Equality (5) shows that, if n is even, the *discrepancy* $A_n - B_n$ is still independent of n; it will, however, depend on our particular choice of sequence $\{u_n\}$ since it will equal $D_1 = u_0^2 - u_1 u_{-1} = u_0^2 - u_1(u_1 - u_0) = u_0^2 + u_0 u_1 - u_1^2$. Thus, the conclusion of our Theorem also holds if n is even, if and only if u_0, u_1 are related by $u_1 = \frac{\sqrt{5}+1}{2} u_0$.

(ii) Since our proof is purely algebraic, it remains valid even if we allow negative values of u_n, provided we interpret area correctly (i.e., allowing for sign). Thus, in particular, we could consider the Fibonacci and Lucas sequences starting with some negative subscript.

(iii) The case considered by Page & Sastry in [2], that is, $u_n = f_n$, does have a special feature of interest. For $f_0^2 + f_0 f_1 - f_1^2 = 1$, so that, in their case, with n even, the area of the region above p_n exceeds that of the region below p_n by exactly one unit. Of course, this phenomenon continues to hold if we take $u_k = f_{n+k}$ *for any even k*. If we take k odd, on the other hand, then, for even values of n, it is the area of the region *below* p_n which exceeds that of the region *above* p_n by one unit.

(iv) Readers will probably wish to refer to [2] for related results, including matrix-generated area-splitting paths.

REFERENCES

1. Walter Ledermann, ed. *Handbook of Applicable Mathematics.* Chichester and New York: John Wiley & Sons, 1980.
2. Warren Page & K. R. S. Sastry. "Area-Bisecting Polygonal Paths." *The Fibonacci Quarterly* **30.3** (1992):263-73.

AMS Classification Numbers: 10A35, 10A40