PIERCE EXPANSIONS AND RULES FOR THE DETERMINATION OF LEAP YEARS

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I. INTRODUCTION

The length of the physical year in days is not an integer. This simple fact has complicated efforts to make a calendar for thousands of years. Both the Julian and the Gregorian calendars use a scheme that involves the periodic insertion of extra days, or *intercalation*. A year with an intercalated day is called a *leap year*.

In the Julian calendar, an extra day was inserted every fourth year. In the Gregorian calendar (commonly in use today) an extra day is inserted every fourth year, exclusive of century years, which are leap years only if divisible by 400. From this, we see that the average length of the year in the Julian calendar was

$$365+\frac{1}{4}=365.25$$

days, while in the Gregorian calendar, the average length is

$$365+\frac{1}{4}-\frac{1}{100}+\frac{1}{400}=365+\frac{1}{4}-\frac{1}{4\cdot 25}+\frac{1}{4\cdot 25\cdot 4}=365.2425$$

days. Both these numbers are approximations to the true length of the year, which is currently about 365.242191 days [1, p. C1].

In this note, we will examine a scheme for leap year determination which generalizes both the Julian and Gregorian calendars and includes the modifications of the Gregorian calendar suggested by McDonnell [2]. Although our results will be phrased in the language of the calendar, they are in fact purely number theoretical in nature.

II. THREE INTERCALATION SCHEMES FOR LEAP YEARS

An *intercalation scheme* describes when to insert extra days in a year to keep the calendar synchronized with the physical year. We assume that exactly 0 or 1 extra days are inserted each year. A year when one day is inserted is called a *leap year*.

Let the length of the year be $I+\beta$ days, where I is an integer and $0\le\beta<1$. Let $L(N)$ count the number of years y in the range $1\le y\le N$ which are declared to be leap years. A good intercalation scheme will certainly have $\lim_{N\to\infty}\frac{L(N)}{N}=\beta$. A much stronger condition is that $|L(N)-\beta N|$ should not be too large.

We now describe three intercalation schemes.

A Method Generalizing the Julian and Gregorian Calendars

Let $a_1, a_2, \ldots$ be a finite or infinite sequence of integers with $a_1\ge 1$ and $a_i\ge 2$ for $i\ge 2$. We call such a sequence (a_i) an *intercalation sequence*.

We now say that N is a leap year if N is divisible by a_1, unless N is also divisible by a_1a_2, in which case it is not, unless N is also divisible by $a_1a_2a_3$, in which case it is, etc. More formally, define the year N to be a leap year if and only if

$$\sum_{k=1}^{\infty}(-1)^{k+1}\operatorname{div}(N, a_1a_2\ldots a_k) = 1,$$

where the function div(x, y) is defined as follows:

$$\operatorname{div}(x, y) = \begin{cases} 1, & \text{if } y|x; \\ 0, & \text{otherwise.} \end{cases}$$

For the Julian calendar, the intercalation sequence is of length 1: $a_1 = 4$. The Gregorian calendar increased the length to 3: $a_1 = 4, a_2 = 25, a_3 = 4$. Herschel ([5], p. 55) proposed extending the Gregorian intercalation sequence by $a_4 = 10$, which results in the estimate $\beta = .24225$. McDonnell [2] has proposed

$$(a_1, a_2, \ldots, a_5) = (4, 25, 4, 8, 27),$$

corresponding to the estimate $\beta = .242199$.

The method has the virtue that it is very easy to remember and is a simple generalization of existing rules. In section III of this paper we examine some of the consequences of this scheme.

An Exact Scheme

Suppose we say that year y is a leap year if and only if

$$\lfloor \beta y + 1/2 \rfloor - \lfloor \beta(y-1) + 1/2 \rfloor = 1.$$

Then $L(N) = \lfloor \beta N + 1/2 \rfloor$; in other words, $L(N)$ is the integer closest to βN. This is clearly the most accurate intercalation scheme possible. However, it suffers from two drawbacks: it is unwieldy for the average person to apply in practice, and β must be known explicitly.

This method can easily be modified to handle the case in which β varies slightly over time. Further, it works well when the fundamental unit is not the year but is, for example, the second. It then describes when to insert a "leap second." This method is essentially that used currently to make yearly corrections to the calendar.

A Method Based on Continued Fractions

We could also find good rational approximations to β using continued fractions. For example, using the approximation .242191 to the fractional part of the solar year, we find

$$.242191 = [0, 4, 7, 1, 3, 17, 5, 1, 1, 7, 1, 1, 2]$$

and the first four convergents are 1/4, 7/29, 8/33, and 31/128. The last convergent, for example, tells us to intercalate 31 days every 128 years. McDonnell notes [personal communication] that had binary arithmetic been in popular use then, Clavius would almost certainly have suggested an intercalation scheme based on this approximation.

The method suffers from the drawback that the method for actually designating the particular years to be leap years is not provided. For example, the third convergent tells us to intercalate 8 days in 33 years, but which of the 33 should be leap years? In 1079, Omar Khayyam suggested that years congruent to 0, 4, 8, 12, 16, 20, 24, and 28 (mod 33) should be leap years [5].

III. SOME THEOREMS

Given an intercalation sequence $(a_1, a_2, \ldots)$, it is easy to compute $L(N)$ using the following theorem.

Theorem 1: Let $L(N)$ be the number of leap years occurring on or before year N, i.e.,

$$L(N) = \sum_{k=1}^{N} \sum_{i \geq 1} (-1)^{i+1} \operatorname{div}(k, a_1 a_2 \ldots a_i).$$

Then

$$L(N) = \sum_{i \geq 1} (-1)^{i+1} \left\lfloor \frac{N}{a_1 a_2 \ldots a_i} \right\rfloor.$$

Proof: It is easy to see that, for $y \geq 1$, we have

$$\operatorname{div}(x, y) = \left\lfloor \frac{x}{y} \right\rfloor - \left\lfloor \frac{x-1}{y} \right\rfloor.$$

Thus, we have

$$\begin{aligned} L(N) &= \sum_{k=1}^{N} \sum_{i \geq 1} (-1)^{i+1} \operatorname{div}(k, a_1 a_2 \ldots a_i) \\ &= \sum_{i \geq 1} (-1)^{i+1} \sum_{k=1}^{N} \operatorname{div}(k, a_1 a_2 \ldots a_i) \\ &= \sum_{i \geq 1} (-1)^{i+1} \sum_{k=1}^{N} \left(\left\lfloor \frac{k}{a_1 \ldots a_i} \right\rfloor - \left\lfloor \frac{k-1}{a_1 \ldots a_i} \right\rfloor \right) \\ &= \sum_{i \geq 1} (-1)^{i+1} \left\lfloor \frac{N}{a_1 \ldots a_i} \right\rfloor, \end{aligned}$$

which completes the proof. □

Theorem 1 explains several things. First of all, it gives the relationship between the intercalation sequence a_i and the length of the physical year in days. Write

$$\alpha = \frac{1}{a_1} - \frac{1}{a_1 a_2} + \cdots.$$

Clearly we have

$$\lim_{N \to \infty} \frac{L(N)}{N} = \alpha.$$

Then if the length of the physical year is $I + \beta$ days, where $0 \leq \beta < 1$, we would like α to be as close as possible to β; for, otherwise, the calendar will move more and more out of synchronization with the physical year.

Therefore, to minimize error, we can assume that the a_i have been chosen so that $\alpha = \beta$. It is somewhat surprising to note that even this choice will cause arbitrarily large differences between the calendar and the physical year; this in spite of the fact that the behavior *on average* will be correct.

Suppose $a_1, a_2, \dots$ have been chosen such that

$$\alpha = \frac{1}{a_1} - \frac{1}{a_1a_2} + \frac{1}{a_1a_2a_3} - \cdots .$$

The next theorem estimates how far out of sync the calendar can be.

Theorem 2: Define $N_j = -1 + a_1 - a_1a_2 + \cdots + (-1)^{j+1}a_1a_2\dots a_j$. Then, for all $r \ge 0$,

$$N_{2r+1}\alpha - L(N_{2r+1}) \ge \sum_{j=1}^{r+1}\left(1-\frac{1}{a_{2j-1}}\right)\left(1-\frac{1}{a_{2j}}\right) \ge \frac{r}{4}.$$

Proof: It is easily verified that, if $i \le j$, then

$$\frac{N_j}{a_1\dots a_i} - \left\lfloor \frac{N_j}{a_1\dots a_i} \right\rfloor = \begin{cases} \frac{N_{i-1}}{a_1\dots a_i}, & \text{if } i \text{ is even;} \\ \frac{N_i}{a_1\dots a_i}, & \text{if } i \text{ is odd.} \end{cases}$$

Thus, we find

$$\begin{aligned} N_{2r+1}\alpha - L(N_{2r+1}) &= \left(N_{2r+1}\sum_{i=1}^{\infty}\frac{(-1)^{i+1}}{a_1a_2\dots a_i}\right) - \sum_{i=1}^{\infty}(-1)^{i+1}\left\lfloor\frac{N_{2r+1}}{a_1\dots a_i}\right\rfloor \\ &= \sum_{i=1}^{2r+2}(-1)^{i+1}\left(\frac{N_{2r+1}}{a_1\dots a_i} - \left\lfloor\frac{N_{2r+1}}{a_1\dots a_i}\right\rfloor\right) + \sum_{i=2r+3}^{\infty}(-1)^{i+1}\frac{N_{2r+1}}{a_1\dots a_i} \\ &\ge \sum_{i=1}^{2r+2}(-1)^{i+1}\left(\frac{N_{2r+1}}{a_1\dots a_i} - \left\lfloor\frac{N_{2r+1}}{a_1\dots a_i}\right\rfloor\right) \\ &= \sum_{j=1}^{r+1}\frac{N_{2j-1}}{a_1\dots a_{2j-1}} - \sum_{j=1}^{r+1}\frac{N_{2j-1}}{a_1\dots a_{2j}} \\ &= \sum_{j=1}^{r+1}\frac{N_{2j-1}}{a_1\dots a_{2j-1}}\left(1-\frac{1}{a_{2j}}\right). \end{aligned}$$

Now, if we observe that $N_{2j-1} \ge a_1\dots a_{2j-1} - a_1\dots a_{2j-2}$, then we find that

$$N_{2r+1}\alpha - L(N_{2r+1}) \ge \sum_{j=1}^{r+1}\left(1-\frac{1}{\alpha_{2j-1}}\right)\left(1-\frac{1}{a_{2j}}\right) \ge \frac{r}{4},$$

which is the desired result. □

Thus, the difference $N_{2r+1}\alpha - L(N_{2r+1})$ can be made as large as desired as $r \to \infty$. Therefore, if α is an irrational number, there is no way to avoid large swings of the calendar.

As an example, consider the Gregorian calendar with intercalation sequence $(a_1, a_2, a_3) = (4, 25, 4)$. Then $N_3 = 303$. For example, in the period from 1600 to 1903, we would expect to see $303 \cdot .2425 = 73.4775$ leap years (assuming the length of the year is precisely 365.2425 days), whereas the Gregorian scheme produces only 72 leap years.

We now assume that the fractional part of the year's length in days is an irrational number α. We also assume that the intercalation sequence a_k is that given by the *Pierce expansion* (see [3], [4], [6]) of α, i.e., the unique way to write

$$\alpha = \frac{1}{a_1} - \frac{1}{a_1 a_2} + \frac{1}{a_1 a_2 a_3} - \cdots$$

such that the a_i are integers with $1 \le a_1 < a_2 \ldots$. It is known that the expansion terminates if and only if α is rational. For example,

$$.242191 = \frac{1}{4} - \frac{1}{4 \cdot 32} + \frac{1}{4 \cdot 32 \cdot 2232} - \frac{1}{4 \cdot 32 \cdot 2232 \cdot 15625}.$$

Then we will show that

Theorem 3: For almost all α, we have

$$\limsup_{N \to \infty} \frac{N\alpha - L(N)}{\sqrt{\log N}} = \frac{\sqrt{2}}{2}.$$

Proof: The proof is in two parts. First, we show that, for all $\varepsilon > 0$, there exists an integer N such that

$$\frac{N\alpha - L(N)}{\sqrt{\log N}} \ge \frac{\sqrt{2}}{2}(1 - 3\varepsilon).$$

Second, we show that, for all $\varepsilon > 0$, we have

$$\frac{N\alpha - L(N)}{\sqrt{\log N}} \le \frac{\sqrt{2}}{2}(1 + 5\varepsilon)$$

for all N sufficiently large.

We need the following two simple lemmas.

Lemma 4: For almost all α,

$$\lim_{n \to \infty} \frac{\log(a_1 \ldots a_n)}{n^2 / 2} = 1.$$

Proof: In [6] it is shown that, for almost all α,

$$\lim_{n \to \infty} \frac{\log a_n}{n} = 1.$$

From this, the desired result follows easily. □

Lemma 5: $\sum_{k=1}^{\infty} \frac{1}{a_k}$ converges for almost all α.

Proof: See Theorem 12 in [6]. □

Now we can return to the proof of the first part of Theorem 3. Let α be chosen, and write

$$C = \frac{1}{a_1} + \frac{1}{a_2} + \cdots .$$

Let ε be given, and choose r_1 sufficiently large so

$$\frac{\log(a_1 \ldots a_r)}{r^2/2} < \frac{1}{(1-\varepsilon)^2}$$

for all $r \ge r_1$. This can be done by Lemma 4. Also choose r sufficiently large so

$$\frac{C\sqrt{2}}{2r+1} < \varepsilon .$$

This can be done by Lemma 5.

Then we find

$$N_{2r+1}\alpha - L(N_{2r+1}) \ge \sum_{j=1}^{r+1}\left(1 - \frac{1}{a_{2j-1}}\right)\left(1 - \frac{1}{a_{2j}}\right) \ge r+1-C. \tag{1}$$

Now, from the definition of N_j, we have $N_{2r+1} \le a_1 \ldots a_{2r+1}$; therefore,

$$\sqrt{\log N_{2r+1}} \le \frac{2r+1}{\sqrt{2}}\left(\frac{1}{1-\varepsilon}\right)$$

because we have chosen r sufficiently large.

Now, dividing both sides of (1) by $\sqrt{\log N_{2r+1}}$ and using the estimate just obtained, we see

$$\begin{aligned}\frac{N_{2r+1}\alpha - L(N_{2r+1})}{\sqrt{\log N_{2r+1}}} &\ge \frac{r+1-C}{2r+1}\sqrt{2}(1-\varepsilon)\\ &\ge \left(\frac{\sqrt{2}}{2} - \frac{C\sqrt{2}}{2r+1}\right)(1-\varepsilon) \ge \left(\frac{\sqrt{2}}{2} - \varepsilon\right)(1-\varepsilon) \ge \frac{\sqrt{2}}{2}(1-3\varepsilon),\end{aligned}$$

which completes the proof of the first part of Theorem 3.

Now let us complete the proof of Theorem 3 by showing that, for almost all α and all N sufficiently large,

$$\frac{N\alpha - L(N)}{\sqrt{\log N}} \le \frac{\sqrt{2}}{2}(1+5\varepsilon).$$

We need the following simple lemma.

Lemma 6: $N\alpha - \sum_{i=1}^{r}(-1)^{i+1}\left\lfloor \frac{N}{a_1 \ldots a_i} \right\rfloor \le \frac{r+1}{2} + \frac{N}{a_1 \ldots a_r(a_r+1)}$.

Proof: Remez [4] has noted that

$$\alpha - \sum_{i=1}^{r}(-1)^{i+1}\frac{1}{a_1 \ldots a_i} \le \frac{1}{a_1 \ldots a_r(a_r+1)} .$$

Multiplying by N, we get

$$N\alpha - N\sum_{i=1}^{r}(-1)^{i+1}\frac{1}{a_1\ldots a_i} \le \frac{N}{a_1\ldots a_r(a_r+1)}. \tag{2}$$

Also we have

$$\left(N\sum_{i=1}^{r}(-1)^{i+1}\frac{1}{a_1\ldots a_i}\right) - \sum_{i=1}^{r}(-1)^{i+1}\left\lfloor\frac{N}{a_1\ldots a_i}\right\rfloor \le \frac{r+1}{2}, \tag{3}$$

since, to maximize this difference, we let the odd-numbered terms equal 1. Adding (2) and (3), we get the desired result. □

Now, given ε, choose N sufficiently large so

(a) $\log N > \frac{1}{2\varepsilon^2}$;

(b) $\frac{\log(a_1\ldots a_r)}{r^2/2} > \frac{1}{(1+\varepsilon)^2}$ for all $r \ge \sqrt{2\log N}$.

By Lemma 4, this can be done for almost all α.

Now, from Lemma 6, we have

$$N\alpha - \sum_{i=1}^{r}(-1)^{i+1}\left\lfloor\frac{N}{a_1\ldots a_i}\right\rfloor \le \frac{r+1}{2} + \frac{N}{a_1\ldots a_r(a_r+1)}. \tag{4}$$

Put $r = \left\lceil \sqrt{2(\log N)(1+\varepsilon)^2}\,\right\rceil$. Then, from part (b) of the hypothesis on N, we have

$$\frac{\log(a_1\ldots a_r)}{(\log N)(1+\varepsilon)^2} > \frac{1}{(1+\varepsilon)^2},$$

and so $a_1\ldots a_r > N$. Therefore

$$\sum_{i=1}^{r}(-1)^{i+1}\left\lfloor\frac{N}{a_1\ldots a_i}\right\rfloor = L(N).$$

Hence, we can substitute in equation (4) to get

$$N\alpha - L(N) \le \frac{r+1}{2} + \frac{1}{a_r+1} \le \frac{\sqrt{2(\log N)(1+\varepsilon)^2}+2}{2} + \frac{1}{a_r+1} \le \frac{\sqrt{2}}{2}\sqrt{\log N}(1+\varepsilon)+2,$$

since $a_r \ge 1$. Dividing both sides of this equation by $\sqrt{\log N}$, we see that

$$\frac{N\alpha - L(N)}{\sqrt{\log N}} \le \frac{\sqrt{2}}{2}(1+\varepsilon) + 2\sqrt{2}\varepsilon \le \frac{\sqrt{2}}{2}(1+5\varepsilon),$$

which completes the proof of Theorem 3. □

In a similar fashion, we can show that

$$\liminf_{N\to\infty}\frac{N\alpha - L(N)}{\sqrt{\log N}} = -\frac{\sqrt{2}}{2}.$$

Roughly speaking, Theorem 3 states that we can expect fluctuations of approximately $\sqrt{\frac{\log N}{2}}$ days at year N of the calendar. Though this type of fluctuation can grow arbitrarily large, it is small for years of reasonable size. For example, for most α, we would have to wait until about the year $3.6 \cdot 10^{42}$ to see fluctuations on the order of a week in size.

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