NUMBER OF MULTINOMIAL COEFFICIENTS NOT DIVISIBLE BY A PRIME

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We consider the n^{th} row of multinomial coefficients of the order ℓ:

$$(j_1, j_2, \ldots, j_\ell) = \frac{n!}{j_1!\, j_2! \ldots j_\ell!},$$

where $j_i \geq 0, i = 1, \ldots, \ell,$ and $n = j_1 + j_2 + \cdots + j_\ell$.

The number of multinomial coefficients not divisible by p^N, where p is prime and N is a fixed whole integer, for various $\ell, p,$ and N was studied by L. Carlitz [1], [2], F. T. Howard [3], [4], [5], [12], R. J. Martin and G. L. Mullen [6], and the author [7]. Let $g(n, \ell, p^N)$ be a number of multinomial coefficients in the n^{th} row of order ℓ not divisible by p^N, and

$$G(n, \ell, p^N) = \sum_{k=1}^{n-1} g(k, \ell, p^N).$$

In the general case, an exact formula for $g(n, \ell, p^N)$ was obtained by F. T. Howard [5] for $N = 1$, 2, 3 and by the author [7] for $N = 1$, 2. It is not difficult to show that the behavior or $g(n, \ell, p^N)$ is very irregular and from that point of view it is better to study $G(n, \ell, p^N)$ which changes more regularly. The function $G(n, \ell, p^N)$ was studied by K. B. Stolarsky [8], [9] and H. Harborth [10] for $N = 1, \ell = p = 2$; by A. H. Stein [11] for $N = 1, \ell = 2$, and arbitrary p; and by the author [7] for arbitrary ℓ and p.

More precisely, the following exact formula was obtained in [7]:

$$G(n, \ell, p) = \sum_{k=0}^{m} (\ell, p-1)^k \frac{a_k}{\ell} \prod_{i=k}^{m} (\ell - 1, a_i), \tag{1}$$

where $n - 1 = a_0 + a_1 p + \cdots + a_m p^m$. It is not difficult to show that $G(n, \ell, p)$ is of the order n^θ, where $\theta = \log_p(\ell, p-1)$. The following theorem gives a more exact result.

Theorem 1: $\alpha \equiv \limsup_{n\to\infty} G(n, \ell, p) / n^\theta = 1$.

Unfortunately, there are no similar results for $\beta \equiv \liminf_{n\to\infty} G(n, \ell, p) / n^\theta$ even in particular cases.

In the general case, only the following elementary estimate is known: $\beta \geq (\ell, p-1)^{-1}$.

In the particular case $p = 2$ (following H. Harborth [10]), we are able to prove the following result.

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Theorem 2: If we consider the sequence $q_r = G(n_r, \ell, 2)/n_r^\theta$ with $n_r = 2n_{r-1} \pm 1, n_0 = 1$, where + or – is chosen so that q_r becomes minimal, then $\{q_r\}$ is strictly decreasing.

This theorem is a generalization of the lemma from [10] for the case of binomial coefficients to the case of multinomial coefficients. We should also note that the sequence $\{n_r\}$ is not the same for different ℓ. In Table 1 the values of n_r for various r and ℓ are given.

TABLE 1

	r											
ℓ	1	2	3	4	5	6	7	8	9	10	15	30
2	3	5	11	21	43	87	173	347	693	1387	44395	1454730075
3	3	7	13	27	53	107	215	429	859	1719	54999	1802202477
4	3	7	13	27	55	109	219	439	877	1755	56171	1840625371
5	3	7	13	27	55	109	219	439	877	1755	56173	1840700855
10	3	7	13	27	55	111	221	443	887	1775	56795	1861082589

In Table 2 we give values of n_r and $q_r = G(n_r, 2, 2)/n_r^\theta$.

TABLE 2

r	n_r	q_r	r	n_r	q_r
1	1	1.000000	26	45460315	0.812556563402
2	3	0.876497	27	90920629	0.812556561634
3	5	0.858126	28	181841259	0.812556559863
4	11	0.827243	29	363682519	0.812556559862
5	21	0.826359	30	727365037	0.812556559272
6	43	0.816719	31	1454730075	0.812556559174
7	87	0.815382	32	2909460149	0.812556559092
8	173	0.813788222	33	5818920299	0.8125565590457850017
9	347	0.813086063	34	11637840597	0.8125565590398820396
10	693	0.812934013	35	23275681195	0.8125565590234059925
11	1387	0.812675296	36	46551362391	0.8125565590216437317
12	2775	0.812657623	37	93102724781	0.8125565590182076960
13	5549	0.812592041	38	186205449563	0.8125565590170475496
14	11099	0.812575228	39	372410899125	0.8125565590166681715
15	22197	0.812567096	40	744821798251	0.8125565590162182798
16	44395	0.812560137	41	1489643596503	0.8125565590162065045
17	88789	0.812559941	42	2979287193005	0.8125565590160702999
18	177579	0.812557589	43	5958574386011	0.8125565590160436690
19	355159	0.812557229	44	11917148772021	0.8125565590160253147
20	710317	0.812556865	45	23834297544043	0.8125565590160134328
21	1420635	0.812556846	46	47668595088085	0.8125565590160123562
22	2841269	0.812556653	47	95337190176171	0.8125565590160082524
23	5682539	0.812556588	48	190674380352343	0.8125565590160076856
24	11365079	0.812556582	49	381348760704685	0.8125565590160069672
25	22730157	0.812556563	50	762697521409371	0.8125565590160066187

On the other hand, if we consider the case $\ell = 2, p = 3, 5, 7$, then there exist increasing sequences $\{n_r\}$ such that $G(n_r, 2, p)/n_r^\theta < G(n_{r-1}, 2, p)/n_{r-1}^\theta$. Calculations give us the following sequences:

$$\begin{aligned} n_0 = 0,\ n_r &= 3n_{r-1}+1, \quad \text{for } p=3,\\ n_0 = 0,\ n_r &= 5n_{r-1}+2, \quad \text{for } p=5,\\ n_0 = 0,\ n_r &= 7n_{r-1}+3, \quad \text{for } p=7. \end{aligned}$$

If we denote $\beta_p = \liminf\limits_{n\to\infty} G(n,2,p)/n^\theta$ and $\hat\beta_p = \liminf\limits_{r\to\infty} G(n_r,2,p)/n_r^\theta$, then

$$\left.\begin{aligned} \hat\beta_3 &= 2^{\log_3 2-1} = 0.774281326315\\ \hat\beta_5 &= 2^{\log_5 3-1} = 0.802518299262\\ \hat\beta_7 &= 2^{\log_7 4-1} = 0.819271977267 \end{aligned}\right\} \tag{$*$}$$

Very probably $\beta_p = \hat\beta_p$, but at the present time we do not have complete proof of this fact. For that purpose it is necessary to show that the sequences $\{n_r\}$ which were defined earlier have the following property: $G(n_r,2,p)/n_r^\theta < \min\limits_{n_{r-1}<n<n_r} G(n,2,p)/n^\theta$, $r=1,2,\ldots$, for $p=2,3,5$, and 7.

Proof of Theorem 1: It follows from (1) that

$$G(p^m,\ell,p)/p^{m\theta} = (\ell,p-1)^m/p^{m\theta} = 1 \tag{2}$$

for all m, which gives us $\alpha \ge 1$.

Furthermore, we will show that

$$b_i \equiv G(ip^m,\ell,p)/(ip^m)^\theta \le 1, \text{ when } 1\le i\le p. \tag{3}$$

For this purpose, we consider the fraction $b_i/b_{i+1}\equiv c_i$ which, due to (1), is

$$c_i = \left(\frac{i+1}{i}\right)^\theta\left(\frac{i}{\ell+1}\right),$$

and we shall show that $c_1 = 2^\theta/(\ell+1)\ge 1$ or, in other words, that

$$(\ell,p-1) > (\ell+1)^{\log_2 p}. \tag{4}$$

Since

$$\frac{(\ell,p-1)(\ell+2)^{\log_2 p}}{(\ell+1)^{\log_2 p}(\ell+1,p-1)} = \frac{\ell+1}{\ell+p}\left(1+\frac{1}{\ell+1}\right)^{\log_2 p},$$

we consider, under $t\ge 3$, the function

$$\varphi(p,t) = \frac{t}{t+p-1}\left(1+\frac{1}{t}\right)^{\log_2 p}$$

as a function of p and, taking the derivitave, we find that

$$\varphi(p,t) = \frac{\varphi(p,t)}{p\ln 2}\left[\ln\left(1+\frac{1}{t}\right) - \frac{p\ln 2}{t+p-1}\right] < \frac{\varphi(p,t)}{tp\ln 2}\left(1-\frac{tp\ln 2}{t+p-1}\right) < 0$$

because $tp/(t+p-1)$ is increased either by p or by t, and

$$\left.\frac{tp\ln 2}{t+p-1}\right|_{t=3,\,p=2} = \frac{3}{2}\ln 2 > 1.$$

Hence, $\varphi(p, \ell+1) < \varphi(2, \ell+1) = 1$, and $(\ell, p-1)/(\ell+1)^{\log_2 p}$ is decreased in ℓ. So

$$(\ell, p-1)/(\ell+1)^{\log_2 p} < (3, p-1)/p^2 \le 1, \text{ when } \ell \ge 3,$$

which proves (4).

As the derivative of the function

$$\psi(x) = \left(\frac{x+1}{x}\right)^\theta \frac{x}{\ell+x}$$

is equal to

$$\psi(x) = \left(\frac{x+1}{x}\right)^\theta \frac{\ell(x+1) - \theta(\ell+x)}{(x+1)(\ell+x)^2},$$

then $\psi(x)$ has only one extreme and, as $c_1 > 1$, this extreme is the minimum. As $b_1 = b_p = 1$ for $2 \le i \le p-1$, we have $b_i \le 1$.

From (1) it is easy to prove, for $0 \le x \le p^m$, that the following recurrent formula is valid:

$$G(a_m p^m + x, \ell, p) = G(a_m p^m, \ell, p) + (\ell-1, a_m) G(x, \ell, p), \tag{5}$$

where $1 \le a_m \le p-1$. We show that

$$G(a_m p^m + x, \ell, p)/(a_m p^m + x)^\theta \le 1 \text{ for all } x = 0, \ldots, p^m - 1,\ m = 0, 1, \ldots \tag{6}$$

is valid.

The inequality (6) is evident when $m = 0$. Let us suppose that (6) is valid in the case of all positive numbers less than m. Then we will have $G(x, \ell, p) \le x^\theta$ for $0 \le x \le p^m$. Then, from (3) and (5), we have

$$\begin{aligned} G(a_m p^m + x, \ell, p)/(a_m p^m + x)^\theta &= [G(a_m p^m, \ell, p) + (\ell-1, a_m) G(x, \ell, p)]/(a_m p^m + x)^\theta \\ &\le [(a_m p^m)^\theta + (\ell-1, a_m)^\theta]/(a_m p^m + x)^\theta \equiv f(x),\ 0 \le x < p^m. \end{aligned} \tag{7}$$

In the interval $[0, p^m]$ the function $f(x)$ has only one extreme, which is the minimum. So (5) is valid. From (3) and (6), we have $\alpha \le 1$ and, when (2) is added, $\alpha = 1$. Theorem 1 is proved.

Proof cf Theorem 2: We suppose

$$G(2n_r + 1, \ell, 1)/(2n_r + 1)^\theta \ge q_r \text{ and } G(2n_r - 1, \ell, 1)/(2n_r - 1)^\theta \ge q_r. \tag{8}$$

If we denote $a = 2n_r$ and $b = \ell^{t_r}/((\ell+1)G(n_r, \ell, 2))$, then, from the definition of q_r and assumptions (7) and (8), we have

$$1 + b \ge \left(1 + \frac{1}{a}\right)^\theta \text{ and } 1 - b \ge \left(1 - \frac{1}{a}\right)^\theta.$$

Addition of these two inequalities yields the contradiction $2 \ge 2 + \theta(\theta-1)/a^2 + \cdots > 2$. Thus, the inequalities (7) cannot both be true, which proves that the sequence $\{q_r\}$ is strictly decreasing. Theorem 2 is proved.

Returning to formulas $(*)$, it is necessary to note that calculations for p = 3, 5, 7 are very simple, by using (1). We omit the proof.

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