DIOPHANTINE REPRESENTATION OF LUCAS SEQUENCES

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1. INTRODUCTION

The Lucas sequences $\{U_n(P,Q)\}$, with parameters P and Q, are defined by $U_0(P,Q)=0$, $U_1(P,Q)=1$, and

$$U_n(P,Q)=PU_{n-1}(P,Q)-QU_{n-2}(P,Q) \quad \text{for } n\geq 2,$$

and the "associated" Lucas sequences $\{V_n(P,Q)\}$ are defined similarly with initial terms equal to 2 and P, for $n=0$ and 1, respectively. The sequences of Fibonacci numbers and Lucas numbers are, of course, $\{F_n\}=\{U_n(1,-1)\}$ and $\{L_n\}=\{V_n(1,-1)\}$.

Several authors (e.g.,[3], [1], [6]) have discussed the conics whose equations are satisfied by pairs of successive terms of the Lucas sequences. In particular, it has been shown that $(x,y)=(w_n,w_{n+1})$ satisfies $y^2-Pxy+Qx^2+eQ^n=0$, where $w_n=U_n(P,Q)$ if $e=-1$ and $w_n=V_n(P,Q)$ if $e=P^2-4Q$. It has apparently not been recognized that the hyperbolas $y^2-Pxy+Qx^2+eR=0$, where $R=1$ if $Q=1$ and $R=\pm1$ if $Q=-1$ *characterize* the Lucas sequences when $e=-1$, and the associated Lucas sequences when $e=P^2-4Q$ is square-free; that is, the set of lattice points on these conics is precisely the set of pairs of consecutive terms of $\{U_n(P,\pm1)\}$ if $e=-1$, and of $\{V_n(P,\pm1)\}$ if $e=P^2-4Q$ is square-free. Accordingly, we shall prove the converse of the results of [3] and [1] by showing that no lattice points exist for the above hyperbolas if $Q=\pm1$ other than (w_n,w_{n+1}) [provided that when $w_n=V_n(P,Q)$, the discriminant D is square-free].

Using the above results, we then construct, for each of the sequences $\{U_n(P,-1)\}$, $\{U_n(P,1)\}$, and $\{V_n(P,1)\}$, a polynomial in two variables of degree 5, and a polynomial of degree 9 for $\{V_n(P,-1)\}$ whose positive values, for positive integral values of the variables, are precisely the terms of the sequence. This extends the results of Jones [4] and [5], who obtained a fifth-degree polynomial whose positive values are the Fibonacci numbers and a ninth-degree polynomial whose positive values are the Lucas numbers.

2. CONICS CHARACTERIZING THE LUCAS SEQUENCES

Assume $P>0$. To simplify notation, we let $U_n=U_n(P,-1)$, $V_n=V_n(P,-1)$, $u_n=U_n(P,1)$, and $v_n=V_n(P,1)$. A proof of the sufficiency in our theorems occurs as a general result in [3]; however, we include an alternate inductive proof in Theorem 1 for completeness.

Theorem 1: Let x and y be positive integers. The pair (x,y) is a solution of

$$y^2-Pxy-x^2=\pm1 \tag{1}$$

iff there exists a positive integer n such that $x=U_n$ and $y=U_{n+1}$.

Proof: We show, first, that $U_{n+1}^2-PU_{n+1}U_n-U_n^2=(-1)^n$, by induction.

If $n=1$, $U_1=1$ and $U_2=P$ and the result clearly holds. Assume $U_n^2-PU_nU_{n-1}-U_{n-1}^2=(-1)^{n-1}$. Then

$$\begin{aligned} U_{n+1}^2 - PU_{n+1}U_n - U_n^2 &= (PU_n + U_{n-1})^2 - P(PU_n + U_{n-1})U_n - U_n^2 \\ &= U_n^2(P^2 - P^2 - 1) + PU_nU_{n-1}(2-1) + U_{n-1}^2 \\ &= -1(U_n^2 - PU_nU_{n-1} - U_{n-1}^2) = (-1)^n. \end{aligned}$$

To see that there are no other solutions of (1) in positive integers, suppose there exist solutions not of the form (U_n, U_{n+1}). Let x be the least positive integer such that, for some positive integer y, (x, y) is a solution of (1) and $(x, y) \neq (U_n, U_{n+1})$ for any positive integer n. Since $(1, P) = (U_1, U_2)$ satisfies (1), $x > 1$. Let $x_0 = y - Px$ and $y_0 = x$. We show that $0 < x_0 < x$ and that (x_0, y_0) satisfies (1). Since $x > 1$, $0 = y^2 - Pxy - x^2 \pm 1 = y(y - Px) - x^2 \pm 1 = yx_0 - x^2 \pm 1$ implies $x_0 > 0$, and from $yx_0 \pm 1 = x^2$, we have $(Px + x_0)x_0 \pm 1 = x^2$, i.e., $Pxx_0 \pm 1 = x^2 - x_0^2$, implying that $x_0 < x$. Now,

$$y_0^2 - Py_0x_0 - x_0^2 = x^2 - Px(y - Px) - (y - Px)^2 = x^2 + Pxy - y^2 = -(\pm 1).$$

Thus, (x_0, y_0) is a solution. By the induction hypothesis, there exists an n such that $x_0 = U_n$ and $y_0 = U_{n+1}$. Then $x = y_0 = U_{n+1}$ and

$$y = Px + x_0 = Py_0 + x_0 = PU_{n+1} + U_n = U_{n+2},$$

contradicting our assumption concerning (x, y).

According to Dickson ([2], Vol. 1, p. 405), Lucas [7] proved that, if x and y are consecutive Fibonacci numbers, then (x, y) is a lattice point on one of the hyperbolas $y^2 - xy - x^2 = \pm 1$, and J. Wasteels [12] proved the converse in 1902.

Theorem 2: Let x and y be positive integers, $x < y$. The pair (x, y) is a solution of

$$y^2 - Pyx + x^2 = 1, \quad P > 2, \tag{2}$$

iff there exists a positive integer n such that $x = u_n$ and $y = u_{n+1}$.

Proof: We note that, because of the symmetry, the assumption that $x < y$ is made without loss of generality. The proof parallels that of Theorem 1. (In proving the necessity, one lets $x_0 = Px - y$ and $y_0 = x$, and easily obtains $x_0 < x$, and $x_0 y = x^2 - 1 < xy \Rightarrow x_0 < x$.)

It is known that, if $D = P^2 + 4$, the general solution in positive integers of $y^2 - Dx^2 = \pm 4$ is $(x, y) = (U_n, V_n)$, and if $D = P^2 - 4$, the general solution of $y^2 - Dx^2 = 4$ is (u_n, v_n). This may be shown using the known general solutions in terms of the fundamental solutions (for example, from $(x_n + y_n\sqrt{D})/2 = [(x_0 + y_0\sqrt{D})/2]^n$ for $x^2 - Dy^2 = 4$; see Mordell [9, p. 55], and Dickson [2, Ch. XIII]). Using Theorems 1 and 2, we provide an alternate derivation of the general solution in terms of Lucas sequences of these Fermat-Pell equations.

Corollary 1: The solutions of $s^2 - Dt^2 = \pm 4$ for $D = P^2 + 4$ and of $s^2 - Dt^2 = 4$ for $D = P^2 - 4$ are precisely the pairs $(t, s) = (U_n, V_n)$ and (u_n, v_n), respectively.

Proof: It is well known that $V_n^2(P, Q) - D \cdot U_n^2(P, Q) = 4Q^n$ [11, p. 44]. Suppose (s, t) is any solution of $s^2 - Dt^2 = \pm 4$ $(D = P^2 + 4)$, i.e., of $s^2 - P^2t^2 = \pm 4 + 4t^2$. It is clear that s and Pt have the same parity, so $y = (s + Pt)/2$ is an integer. Upon substituting for s,

$$(2y-Pt)^2 - P^2t^2 = \pm 4 + 4t^2 \Rightarrow 4y^2 - 4Pty = \pm 4 + 4t^2.$$

That is, $y^2 - Pyt - t^2 = \pm 1$. By Theorem 1, $y = U_{n+1}$ and $t = U_n$ for some n. Now it is known that $V_n(P, Q) = 2U_{n+1}(P, Q) - PU_n(P, Q)$ [11, p. 44], implying that $s = V_n$.

The proof of the necessity for $s^2 - Dt^2 = 4$, $D = P^2 - 4$ is similar.

3. CONICS CHARACTERIZING THE ASSOCIATED LUCAS SEQUENCES

It is interesting that the solutions of the hyperbolas $y^2 - Pxy - x^2 = \pm D$, for $D = P^2 + 4$, include (V_n, V_{n+1}) for $n \geq 0$, and the solutions of $y^2 - Pxy + x^2 = -D$, for $D = P^2 - 4$, include (v_n, v_{n+1}) for $n \geq 0$ [3], but that there may be, in general, additional pairs of integral solutions. A case in point: $y^2 - 4xy - x^2 = 20$ has $(x, y) = (1, 7)$ as a solution (but $V_n \neq 1$ for any $n \geq 0$). It may be shown, however, that there are no additional solutions if D is square-free.

Theorem 3: Let $P^2 + 4 = D = a^2 d$, d square-free. The set of lattice points with positive coordinates on the hyperbolas

$$y^2 - Pyx - x^2 = \pm D \tag{3}$$

is precisely the set $\{(V_n, V_{n+1})\}$ $(n \geq 0)$ iff the sets of x-coordinates of the solution sets of $x^2 - Dy^2 = \pm 4$ and $x^2 - dz^2 = \pm 4$ are equal.

Proof: As remarked above, (V_n, V_{n+1}) satisfies (3) for all $n \geq 0$. Assume that $x, y > 0$ and (x, y) is a solution of (3). Now, since P and D have the same parity, (3) implies that

$$y = \left[Px + \sqrt{D(x^2 \pm 4)}\right] / 2 = \left[Px + a\sqrt{d(x^2 \pm 4)}\right] / 2$$

is an integer iff $d(x^2 \pm 4)$ is a square; that is, iff, for some integer z, $x^2 \pm 4 = dz^2$, i.e., $x^2 - dz^2 = \pm 4$. Thus, the set of lattice points on (3) is precisely the set $\{V_n, V_{n+1}\}$ iff $x = V_n$ for some $n \geq 0$. By Corollary 1, on the other hand, the pair (x, y) is a solution of $x^2 - Dy^2 = \pm 4$ iff $x = V_n$ for some $n \geq 0$. This proves the theorem.

If D is square-free, then $d = D$, and we immediately have

Corollary 2: Let x and y be positive integers, and $D = P^2 + 4$ be square-free. The pair (x, y) is a solution of $y^2 - Pxy - x^2 = \pm D$ iff there exists a nonnegative integer n such that $x = V_n$ and $y = V_{n+1}$.

We note that the equations $x^2 - Dy^2 = \pm 4$ and $x^2 - dz^2 = \pm 4$ of Theorem 3 may have solution sets having identical x-coordinates when $D \neq d$. For example, if $D = 4d$ and $d \equiv 2$ or 3 (mod 4), since in these cases z must be even.

We may establish, in exactly the same way as for Theorem 3, the corresponding theorem for $y^2 - Pyx + x^2 = -D$, with $D = P^2 - 4$. We state only the analogous corollary.

Corollary 3: Let $D = P^2 - 4$ be square-free and x and y be positive integers. The pair (x, y) is a solution of

$$y^2 - Pyx + x^2 = -D \tag{4}$$

if there exists a nonnegative integer n such that $x = v_n$ and $y = v_{n+1}$.

4. DIOPHANTINE REPRESENTATION OF THE SEQUENCES

The set of terms of any Lucas sequence is a recursively enumerable set, and such sets have been shown to be Diophantine [8]. That is, for each recursively enumerable set S, there exists a polynomial $\mathcal{P}$ with integral coefficients, in variables $x_1, \ldots, x_n$, such that $x \in S$ iff there exist positive integers $y_1, \ldots, y_{n-1}$ such that $\mathcal{P}(x, y_1, \ldots, y_{n-1}) = 0$. As a consequence, it is possible to construct a polynomial whose positive values are precisely the elements of S. The construction is due to Putnam [10], who observed that $x(1-\mathcal{P}^2)$ has the desired property. Using equations (1), (2), (3), (4), and Corollary 1, we now obtain such polynomials for the set of terms of the sequences $\{U_n(P,-1)\}$, $\{U_n(P,1)\}$, $\{V_n(P,-1)\}$, and $\{V_n(P,1)\}$.

Theorem 5: Let $\mathcal{U}(P,Q)$ denote the set of terms of the sequence $\{U_n(P,Q)\}$, and $\mathcal{V}(P,Q)$ denote the set of terms of the sequence $\{V_n(P,Q)\}$. Then, if x and y assume all positive integral values, the set S is identical to the set of positive values of the polynomial

(i) $x[2-(y^2-Pxy-x^2)^2]$ if $S = \mathcal{U}(P,-1)$,

(ii) $x[2-(y^2-Pxy+x^2)^2]$ if $S = \mathcal{U}(P,1)$, $P>2$,

(iii) $y[1-((y^2-Dx^2)^2-16)^2]$ if $S = \mathcal{V}(P,-1)$, $D = P^2+4$,

(iv) $y[1-((y^2-Dx^2)-4)^2]$ if $S = \mathcal{V}(P,1)$, $D = P^2-4$.

Proof: In view of Theorems 1 and 2 and Corollary 1, the proof is obvious, provided we show that $y^2-Pxy-x^2$ and $y^2-Pxy+x^2$ $(P>2)$ are never 0 for x and y integers. However, if either equals 0, then

$$y = \frac{Px \pm x\sqrt{P^2+4}}{2} \quad \text{or} \quad y = \frac{Px \pm x\sqrt{P^2-4}}{2}, \quad (P>2);$$

clearly, since $D = P^2 \pm 4$ is not a square, y is irrational for all integral x values.

By Corollary 1, the polynomials in (i) and (ii) may be given, alternatively, as

$$x\left[1-\left((y^2-Dx^2)^2-16\right)^2\right], \text{ for } D = P^2+4,$$

and

$$x\left[1-\left((y^2-Dx^2)-4\right)^2\right], \text{ for } D = P^2-4,$$

respectively. And, by Corollaries 2 and 3, the polynomials in (iii) and (iv) may be given, alternatively, if D is square-free, as

$$x\left[1-\left((y^2-Pyx-x^2)^2-(P^2+4)^2\right)^2\right]$$

and

$$x\left[1-(y^2-Pxy+x^2+P^2-4)^2\right],$$

respectively; however, in case (i) of the theorem, the degree of the alternative is higher.

For a summary of results on polynomials representing various additional sets, we refer the reader to [11, Ch. 3, III].

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NEW EDITORIAL POLICIES

The Board of Directors of The Fibonacci Association during their last business meeting voted to incorporate the following two editorial policies effective January 1, 1995.

1. All articles submitted for publication in *The Fibonacci Quarterly* will be blind refereed.
2. In place of Assistant Editors, *The Fibonacci Quarterly* will change to utilization of an Editorial Board.