A GENERALIZATION OF THE CATALAN IDENTITY AND SOME CONSEQUENCES

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1. INTRODUCTION

The Catalan identity

$$F_{n-r}F_{n+r} - F_n^2 = (-1)^{n-r+1}F_r^2 \tag{1.1}$$

has several generalizations. Here we obtain a new generalization and use it to generalize the Gelin-Cesàro identity

$$F_n^4 - F_{n-2}F_{n-1}F_{n+1}F_{n+2} = 1, \tag{1.2}$$

which was stated by Gelin and proved by Cesàro (see [1], p. 401). Furthermore, we establish that a certain expression arising from three-term recurrence relations is a perfect square, and this generalizes previous work.

Using the notation of Horadam [2], let

$$W_n = W_n(a, b; p, q) \tag{1.3}$$

so that

$$W_n = pW_{n-1} - qW_{n-2},\ W_0 = a,\ W_1 = b,\ n \geq 2. \tag{1.4}$$

If α, β, assumed distinct, are the roots of

$$\lambda^2 - p\lambda + q = 0, \tag{1.5}$$

we have the Binet form [2]

$$W_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta}, \tag{1.6}$$

in which

$$\begin{cases} A = b - a\beta \\ B = b - a\alpha. \end{cases} \tag{1.7}$$

Write

$$e = pab - qa^2 - b^2 = -AB. \tag{1.8}$$

As usual, $U_n = W_n(0, 1; p, q)$ is the fundamental sequence of Lucas [4].

2. THE MAIN RESULT

We now generalize the Catalan identity and obtain some consequences.

Theorem: For $W_n = W_n(a, b; p, q)$ and $Y_n = W_n(a_1, b_1; p, q)$,

$$W_nY_{n+r+s} - W_{n+r}Y_{n+s} = \Psi(s)q^nU_r, \tag{2.1}$$

where

$$\Psi(s) = (pa_1b - qaa_1 - bb_1)U_s + (ab_1 - a_1b)U_{s+1}.$$

Proof: Using the Binet forms for W_n and Y_n we obtain, after some algebra,

$$W_nY_{n+r+s} - W_{n+r}Y_{n+s} = \frac{(AB_1\beta^s - A_1B\alpha^s)q^nU_r}{\alpha-\beta},$$

where, in the Binet form for Y_n,

$$\begin{cases} A_1 = b_1 - a_1\beta, \\ B_1 = b_1 - a_1\alpha. \end{cases} \tag{2.2}$$

Now, using (1.7) and (2.2) we see, after simplifying, that $\frac{AB_1\beta^s - A_1B\alpha^s}{\alpha-\beta}$ reduces to $\Psi(s)$. □

In (2.1), replacing n by $n-r$ and s by r gives

$$W_{n-r}Y_{n+r} - W_nY_n = \Psi(r)q^{n-r}U_r. \tag{2.3}$$

Replacing r by $r+1$ in (2.3), we have

$$W_{n-r-1}Y_{n+r+1} - W_nY_n = \Psi(r+1)q^{n-r-1}U_{r+1}. \tag{2.4}$$

Adding (2.3) and (2.4) gives

$$W_{n-r}Y_{n+r} + W_{n-r-1}Y_{n+r+1} = 2W_nY_n + \Psi(r)q^{n-r}U_r + \Psi(r+1)q^{n-r-1}U_{r+1}. \tag{2.5}$$

Subtracting (2.4) from (2.3) gives

$$W_{n-r}Y_{n+r} - W_{n-r-1}Y_{n+r+1} = \Psi(r)q^{n-r}U_r - \Psi(r+1)q^{n-r-1}U_{r+1}. \tag{2.6}$$

Squaring (2.5) and subtracting the square of (2.6), we obtain

$$\begin{aligned} W_{n-r-1}W_{n-r}Y_{n+r}Y_{n+r+1} = W_n^2Y_n^2 &+ W_nY_nq^{n-r-1}(q\Psi(r)U_r + \Psi(r+1)U_{r+1}) \\ &+ \Psi(r)\Psi(r+1)q^{2n-2r-1}U_rU_{r+1}. \end{aligned} \tag{2.7}$$

Putting $r=1$ in (2.7) yields

$$W_{n-2}W_{n-1}Y_{n+1}Y_{n+2} = W_n^2Y_n^2 + W_nY_nq^{n-2}(q\Psi(1) + p\Psi(2)) + p\Psi(1)\Psi(2)q^{2n-3}. \tag{2.8}$$

In (2.1), substituting $r=-1$, $s=m-n+1$ and noting that $U_{-1} = -q^{-1}$, we obtain

$$W_nY_m - W_{n-1}Y_{m+1} = -\Psi(m-n+1)q^{n-1}. \tag{2.9}$$

Furthermore, if $n = m-1$, then (2.9) yields

$$W_{m-1}Y_m - W_{m-2}Y_{m+1} = -\Psi(2)q^{m-2}. \tag{2.10}$$

Finally, from (2.1), it follows that

$$(W_nY_{n+r+s} - W_{n+r}Y_{n+s})^2 = \Psi^2(s)q^{2n}U_r^2,$$

so that

$$4W_nW_{n+r}Y_{n+s}Y_{n+r+s} + \Psi^2(s)q^{2n}U_r^2 = (W_nY_{n+r+s} + W_{n+r}Y_{n+s})^2,$$

thus establishing that

$$4W_nW_{n+r}Y_{n+s}Y_{n+r+s}+\Psi^2(s)q^{2n}U_r^2 \quad (2.11)$$

is a perfect square for nonnegative integers n, r, s and integers a, b, a_1, b_1, p, q.

3. RELATION TO OTHER GENERALIZATIONS

The results of the previous section generalize results of Horadam and Shannon [3] who, in turn, generalized work of Morgado [5] on the Fibonacci numbers. It suffices then to indicate how our work generalizes that of Horadam and Shannon.

In (2.1), when $(a_1, b_1) = (a, b)$, we have $\{W_n\} = \{Y_n\}$ and $\Psi(s) = eU_s$, so that (2.1) becomes

$$W_nW_{n+r+s} - W_{n+r}W_{n+s} = eq^nU_rU_s,$$

which Horadam and Shannon gave as a generalization of the Catalan identity. Under the same circumstances, noting that $\Psi(1) = e$ and $\Psi(2) = ep$, (2.8) reduces to

$$W_{n-2}W_{n-1}W_{n+1}W_{n+2} = W_n^4 + W_n^2eq^{n-2}(p^2+q) + e^2q^{2n-3}p^2,$$

which Horadam and Shannon gave as a generalization of the Gelin-Cesàro identity.

Similarly, (2.9) and (2.10) reduce, respectively, to

$$W_nW_m - W_{n-1}W_{m+1} = -eq^{n-1}U_{m-n+1}$$

and

$$W_nW_{n-1} - W_{n-2}W_{n+1} = -epq^{n-2},$$

which are generalizations of results for Fibonacci numbers due to D'Ocagne (see [1], p. 402).

Finally, the expression (2.11) reduces to

$$4W_nW_{n+r}W_{n+s}W_{n+r+s} + e^2q^{2n}U_r^2U_s^2,$$

which was proved by Horadam and Shannon to be a perfect square.

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