# A DIFFERENCE-OPERATIONAL APPROACH TO THE MÖBIUS INVERSION FORMULAS

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## 1. INTRODUCTION

Worth noticing is that the well-known Möbius inversion formulas in the elementary theory of numbers (cf. e.g., [2] and [3]),

$$f(n)=\sum_{d|n} g(d) \tag{1}$$

and

$$g(n)=\sum_{d|n} f(d)\mu(n/d)=\sum_{d|n} f(n/d)\mu(d), \tag{2}$$

may be viewed precisely as a discrete analog of the following Newton-Leibniz fundamental formulas

$$F(x_1,\dots,x_s)=\int_{c_1}^{x_1}\cdots\int_{c_s}^{x_s} G(t_1,\dots,t_s)dt_s\dots dt_1 \tag{3}$$

and

$$G(x_1,\dots,x_s)=\frac{\partial}{\partial x_1}\cdots\frac{\partial}{\partial x_s}F(x_1,\dots,x_s), \tag{4}$$

wherein the summations of (1) and (2) are taken over all the divisors $d$ of $n$, and $G(t_1,\dots,t_s)$ is an integrable function so that $F(x_1,\dots,x_s)=0$ when there is some $x_i=c_i$ $(1\le i\le s)$. This will be made clear in what follows.

Let us use the prime factorization forms for $n$ and $d$, say $n=p_1^{x_1}\cdots p_s^{x_s}$ and $d=p_1^{t_1}\cdots p_s^{t_s}$, $p_i$ being distinct primes, $x_i$ and $t_i$ being nonnegative integers with $0\le t_i\le x_i$ $(i=1,\dots,s)$, and replace $f(n)$ and $g(d)$ of (1) by $f((x))\equiv f(x_1,\dots,x_s)$ and $g((t))\equiv g(t_1,\dots,t_s)$, respectively. Then one may rewrite (1) and (2) as multiple sums of the following:

$$f(x_1,\dots,x_s)=\sum_{0\le t_i\le x_i} g(t_1,\dots,t_s) \tag{5}$$

and

$$g(x_1,\dots,x_s)=\sum_{0\le t_i\le x_i} f(x_i-t_1,\dots,x_s-t_s)\mu_1(t_1,\dots,t_s), \tag{6}$$

where each summation is taken over all the integers $t_i$ $(i=1,\dots,s)$ such that $0\le t_i\le x_i$, and $\mu_1((t))\equiv\mu_1(t_1,\dots,t_s)$ is defined by

$$\mu_1((t))=\begin{cases}(-1)^{t_1+\cdots+t_s}, & \text{if all } t_i\le 1,\\ 0, & \text{if there is a } t_i\ge 2.\end{cases} \tag{7}$$

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Evidently $\mu_1((t)) = \mu(d)$ is just the classical Möbius function defined for positive integers $d$ with $\mu(1) = 1$ (cf. [4]).

Now we introduce the backward difference operator $\underset{x}{\Delta}$ and its inverse $\underset{x}{\Delta}^{-1}$ by the following:

$$\underset{x}{\Delta} f(x) = f(x) - f(x-1),\ \underset{x}{\Delta}^{-1} g(x) = \sum_{0 \le t \le x} g(t) \tag{8}$$

so that $\underset{x}{\Delta}\underset{x}{\Delta}^{-1} g(x) = g(x)$, $\underset{x}{\Delta}^{-1}\underset{x}{\Delta} f(x) = f(x)$, and we may denote $\underset{x}{\Delta}\underset{x}{\Delta}^{-1} = \underset{x}{\Delta}^{-1}\underset{x}{\Delta} = I$ with $If(x) \equiv f(x)$, where we assume that $f(x) = g(x) = 0$ for $x < 0$. Thus, (5) and (6) can be expressed as

$$f((x)) = \underset{x_1}{\Delta}^{-1} \cdots \underset{x_s}{\Delta}^{-1} g((x)) \tag{9}$$

and

$$g((x)) = \underset{x_1}{\Delta} \cdots \underset{x_s}{\Delta} f((x)), \tag{10}$$

where it is always assumed that $f((x)) = g((x)) = 0$ whenever there is some $x_i < 0$ $(1 \le i \le s)$, $s$ being any positive integer.

Apparently, the reciprocal pair (9) $\Leftrightarrow$ (10) is just a discrete analog of the inverse relations (3) $\Leftrightarrow$ (4). This is what we claimed in the beginning of this section.

## 2. A GENERALIZATION OF (9) $\Leftrightarrow$ (10)

Difference operators of higher orders may be defined inductively as follows:

$$\underset{x}{\Delta}^r = \underset{x}{\Delta}\underset{x}{\Delta}^{r-1},\ \underset{x}{\Delta}^{-r} = \underset{x}{\Delta}^{-1}\underset{x}{\Delta}^{-(r-1)},\ (r \ge 2),\ \Delta^0 = I.$$

***Lemma 1:*** For any positive integer $r$, we have $\underset{x}{\Delta}^r\underset{x}{\Delta}^{-r} = \underset{x}{\Delta}^{-r}\underset{x}{\Delta} = I$.

***Proof:*** (By induction.) The case $r = 1$ has been noted previously. If it holds for the case $r = k \ge 1$, then, for any given $f(x)$,

$$\underset{x}{\Delta}^{k+1}\underset{x}{\Delta}^{-k-1} f(x) = \underset{x}{\Delta}^{k}\underset{x}{\Delta}\underset{x}{\Delta}^{-1}\underset{x}{\Delta}^{-k} f(x) = \underset{x}{\Delta}^{k} I \underset{x}{\Delta}^{-k} f(x) = \underset{x}{\Delta}^{k}\underset{x}{\Delta}^{-k} f(x) = f(x),$$

and, consequently, $\underset{x}{\Delta}^{k+1}\underset{x}{\Delta}^{-(k+1)} = I$. Hence, $\underset{x}{\Delta}^r\underset{x}{\Delta}^{-r} = I$ holds for all $r \ge 1$. Similarly, $\underset{x}{\Delta}^{-r}\underset{x}{\Delta}^r = I$ may also be verified by induction. □

In what follows, we always assume that every function $f((x))$ or $g((x))$ will vanish whenever there is some $x_i < 0$ $(1 \le i \le s)$.

***Lemma 2:*** For every given $(r) \equiv (r_1, \dots, r_2)$ with $r_i \ge 1$, we have the following pair of reciprocal relations:

$$f((x)) = \left(\prod_{i=1}^{s} \underset{x_i}{\Delta}^{-r_i}\right) g((x)) \tag{11}$$

and

$$g((x)) = \left(\prod_{i=1}^{s} \underset{x_i}{\Delta}^{r_i}\right) f((x)). \tag{12}$$

***Proof:*** This is easily verified by repeated application of Lemma 1. In fact, the implication $(11) \Rightarrow (12)$ follows from the identity

$$\left(\prod_{i=1}^{s} \Delta_{x_i}^{r_i}\right)\left(\prod_{i=1}^{s} \Delta_{x_i}^{-r_i}\right) = I. \tag{13}$$

Similarly, we have $(12) \Rightarrow (11)$. □

Evidently, the reciprocal pair $(11) \Leftrightarrow (12)$ implies $(1) \Leftrightarrow (2)$ with $r_i = 1$ $(i = 1, \dots, s)$, since (1) and (2) are equivalent to (9) and (10), respectively.

## 3. AN EXPLICIT FORM

It is not difficult to find some explicit expressions for the right-hand sides of (11) and (12). For the case $s = 1$, write $f((x)) = f(x)$. By mathematical induction, we easily obtain, for $r \geq 2$,

$$\Delta_x^r f(x) = \sum_{0 \leq t \leq r} (-1)^t \binom{r}{t} f(x-t), \tag{14}$$

$$\Delta_x^{-r} g(x) = \sum_{0 \leq t \leq t_1 \leq \cdots \leq t_{r-1} \leq x} g(t) \tag{15}$$

$$\Delta_x^{-r} g(x) = \sum_{0 \leq t \leq x} \binom{x-t+r-1}{r-1} g(t) = \sum_{0 \leq t \leq x} \binom{t+r-1}{r-1} g(x-t), \tag{16}$$

where the summation contained in (15) is taken over all the $r$-tuples of integers $(t, t_1, \dots, t_{r-1})$ such that $0 \leq t \leq t_1 \leq \cdots \leq t_{r-1} \leq x$. It is readily seen that, for each fixed $t \geq 0$, the number of all such $r$-tuples is given by $\binom{x-t+r-1}{r-1}$, so that (16) follows from (15).

As may be verified, the explicit forms given by (14) and (16) can be used to produce another proof of Lemma 1 and of Lemma 2, with the aid of the combinatorial identity

$$\sum_{j=0}^{r} (-1)^j \binom{r}{j} \binom{n-j+r-1}{r-1} = \begin{cases} 1 & \text{when } n = 0, \\ 0 & \text{when } n \geq 1. \end{cases}$$

Actually, this identity follows at once from comparing the coefficients of $z^n$ on both sides of the product of the following expansions:

$$(1-z)^r = \sum_{j \geq 0} (-1)^j \binom{r}{j} z^j, \quad (1-z)^{-r} = \sum_{j \geq 0} \binom{j+r-1}{r-1} z^j.$$

In what follows, we denote $(x) - (t) \equiv (x_1 - t_1, \dots, x_s - t_s)$ with $(x) \equiv (x_1, \dots, x_s)$ and $(t) \equiv (t_1, \dots, t_s)$ as before. Also, we use $(0) \leq (t) \leq (x)$ to denote the conditions $0 \leq t_i \leq x_i$ $(i = 1, \dots, s)$, etc. As the right-hand sides of (11) and (12) consist of only repeated sums, we see that Lemma 2 together with (14) and (16) for $r = r_i$, $x = x_i$ $(i = 1, \dots, s)$ imply the following

***Theorem:*** For any given $(r) \equiv (r_1, \dots, r_s)$ with all $r_i \geq 1$, there hold the reciprocal relations

$$f((x)) = \sum_{(0) \leq (t) \leq (x)} \mu_{(r)}^{-1}((t)) g((x)-(t)) \tag{17}$$

and

$$g((x)) = \sum_{(0)\le(t)\le(r)} \mu_{(r)}((t))f((x)-(t)), \tag{18}$$

where $\mu_{(r)}((t))$ and $u_{(r)}^{-1}((t))$ are defined by the following:

$$\mu_{(r)}((t)) = \prod_{i=1}^{s}\binom{r_i}{t_i}(-1)^{t_i}, \quad \mu_{(r)}^{-1}((t)) = \prod_{i=1}^{s}\binom{t_i+r_i-1}{r_i-1}. \tag{19}$$

Note that for the case $(r)\equiv(1,\dots,1)$ the function $\mu_{(r)}((t))$ becomes the ordinary Möbius function, so that (17) and (18) constitute a generalized pair of Möbius inversions. Accordingly, $\mu_{(r)}^{-1}((t))$ may be called the inverse Möbius function with given $(r)\equiv(r_1,\dots,r_s)$ as a parametric vector. Moreover, it may be observed that the condition $(0)\le(t)\le(r)$ under the summation of (18) may also be replaced by $(0)\le(t)\le(x)$ inasmuch as $g((x)-(t))=0$ whenever there is some $x_i-t_i<0$. Consequently, (17) and (18) may be expressed as "convolutions":

$$f((x)) = \mu_{(r)}^{-1} * g((x)), \quad g((x)) = \mu_{(r)} * f((x)). \tag{20}$$

***Remark:*** Reversing the ordering relations in the summation process, one may find that there are dual forms corresponding to (17) and (18). Suppose that $(m)\equiv(m_1,\dots,m_s)$ is a fixed $s$-tuple of positive integers and that we are considering such functions $f^*((x))$ and $g^*((x))$ with the property that $f^*((x))=g^*((x))=0$ whenever there is some $x_i>m_i$ $(1\le i\le s)$. Then the dual forms of (17)–(18) are given by

$$f^*((x)) = \sum_{(x)\le(t)\le(m)} \mu_{(r)}^{-1}((t)-(x))g^*((t)) \tag{21}$$

and

$$g^*((x)) = \sum_{(x)\le(t)\le(m)} \mu_{(r)}((t)-(x))f^*((t)), \tag{22}$$

where the summations are taken over all $(t)$ such that $x_i\le t_i\le m_i$ $(i=1,\dots,s)$. This reciprocal pair $(21)\Leftrightarrow(22)$ has certain applications to the Probability Theory of Arbitrary Events. For instance, the case $(r)\equiv(1,\dots,1)$ may be used to yield a generalization of Poincaré's formula for the calculus of probabilities (cf. [1]).

## 4. A CONSEQUENCE OF THE THEOREM

Returning now to the theory of numbers, let us denote by $\partial(p|d)$ the highest power of the prime number $p$ that divides $d$. Thus, for $d=p_1^{t_1}\cdots p_s^{t_s}$, we have $\partial(p_i|d)=t_i$. Also, we define $\partial(1|d)=0$.

Notice that the functions $f(n)=f(p_1^{x_1}\cdots p_s^{x_s})$ and $g(d)=g(p_1^{t_1}\cdots p_s^{t_s})$ may be mapped to the corresponding functions $\tilde{f}((x))$ and $\tilde{g}((t))$, respectively. Thus, making use of the theorem with $r_i=r$ $(i=1,\dots,s)$, we easily get a pair of reciprocal relations, as follows,

$$f(n) = \sum_{d|n} g\left(\frac{n}{d}\right)\nu_r(d) \tag{23}$$

and

$$g(n)=\sum_{d|n} f\left(\frac{n}{d}\right)\mu_r(d), \tag{24}$$

where $\nu_r(d)$ and $\mu_r(d)$ are defined by the following:

$$\nu_r(d)=\prod_{p|d}\binom{\partial(p|d)+r-1}{r-1},\quad \mu_r(d)=\prod_{p|d}\binom{r}{\partial(p|d)}(-1)^{\partial(p|d)}.$$

Obviously, the classical pair (1)–(2) is a particular case of (23)–(24) with $r=1$. Moreover, for the case $r=2$, we have

$$\nu_2(d)=\prod_{p|d}(\partial(p|d)+1)\overset{\text{def}}{=}\delta(d),$$

where $\delta(d)$ stands for the divisor function that represents the number of divisors of $d$. Consequently, (23)–(24) imply the following reciprocal pair as the second interesting case:

$$f(n)=\sum_{d|n} g\left(\frac{n}{d}\right)\delta(d); \tag{25}$$

$$g(n)=\sum_{d|n} f\left(\frac{n}{d}\right)\mu_2(d). \tag{26}$$

Surely (25)–(26) may be used to obtain various relations between special number sequences by taking $g(n)$ or $f(n)$ to be special number-theoretic functions.

## REFERENCES


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