SUMS OF ARITHMETIC PROGRESSIONS

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Readers of *Mathematical Spectrum* have recently indicated an interest in the problem of expressing a natural number n as a sum of (at least two) consecutive natural numbers (see [1]-[5]). Bob Bertuello showed that this is not possible when n is a power of 2. We prove here a general result which determines those natural numbers that can be expressed as a sum of natural numbers in arithmetic progression with common difference d.

A consequence of the case $d=1$ is that a natural number is a sum of consecutive natural numbers if and only if it is not a power of 2. (We believe this result may already be known, but have not been able to trace it in the literature.) When $d=2$, our theorem shows that a natural number n is a sum of natural numbers in arithmetic progression with common difference 2 if and only if n is not a prime. We shall also illustrate the result for the case $d=3$.

Theorem: Let the natural number d be given. Then the natural number $n=2^h m$. where m is odd and $n>1$, is a sum of natural numbers which form an arithmetic progression with common difference d if and only if

(1) for d odd, n is not a power of 2 and either $m>d(2^{h+1}-1)$ or $n>\frac{1}{2}dp(p-1)$, where p is the smallest odd prime factor of n,

(2) for d even, either n is even and $n>d$ or n is odd and $n>\frac{1}{2}dp(p-1)$, where again p is the smallest odd prime factor of n.

Proof: We first prove that the conditions given are necessary. Suppose that n is a sum of natural numbers which form an arithmetic progression with common difference d, say,

$$n=r+(r+d)+(r+2d)+\cdots+(r+sd)$$

for some natural numbers r and s. (It is always understood that there is more than one term in the sum.) Then

$$n=(s+1)\left(r+\frac{sd}{2}\right).$$

We consider four cases.

Case 1. d odd, s odd. Then

$$n=\frac{s+1}{2}(2r+sd)$$

and $2r+sd$ is an odd divisor of n. Hence, n is not a power of 2 and $\frac{s+1}{2}\geq 2^h$, i.e., $s\geq 2^{h+1}-1$. Thus,

$$2^h m=n>\frac{s+1}{2}sd\geq 2^h d(2^{h+1}-1),$$

whence $m>d(2^{h+1}-1)$.

Case 2. d odd, s even. Then

$$n=(s+1)\left(r+\frac{s}{2}d\right)$$

and n has the odd divisor $s+1>1$. Hence, n is not a power of 2 and $s+1\geq p$, where p is the smallest odd prime factor of n. Thus,

$$n>(s+1)\frac{s}{2}d\geq\frac{1}{2}dp(p-1).$$

Case 3. d even, n even. Clearly, $n>d$.

Case 4. d even, n odd. Then

$$n=(s+1)\left(r+s\frac{d}{2}\right)$$

and n has the (odd) divisor $s+1>1$. The argument of Case 2 gives $n>\tfrac{1}{2}dp(p-1)$.

We now prove that the conditions are sufficient. Again, there are four cases.

Case 1. d odd, $m>d(2^{h+1}-1)$ (so that n is not a power of 2). Put $s=2^{h+1}-1$ and $r=\tfrac{1}{2}[m-d(2^{h+1}-1)]$. Then r and s are natural numbers and

$$\begin{aligned}r+(r+d)+(r+2d)+\cdots+(r+sd)&=(s+1)\left(r+\frac{1}{2}sd\right)\\&=2^{h+1}\left\{\frac{1}{2}[m-d(2^{h+1}-1)]+\frac{1}{2}d(2^{h+1}-1)\right\}\\&=2^{h}m=n.\end{aligned}$$

It is worth noting that, in this case, the arithmetic progression contains $s+1=2^{h+1}$ terms.

Case 2. d odd, n not a power of 2, and $n>\tfrac{1}{2}dp(p-1)$. Choose $s=p-1$ and $r=n/p-\tfrac{1}{2}d(p-1)$. Then r and s are natural numbers and

$$\begin{aligned}r+(r+d)+(r+2d)+\cdots+(r+sd)&=(s+1)\left(r+\frac{1}{2}sd\right)\\&=p\left\{\frac{n}{p}-\frac{1}{2}d(p-1)+\frac{1}{2}d(p-1)\right\}=n.\end{aligned}$$

In this case, the arithmetic progression contains p terms, where p is the smallest odd prime factor of n.

Case 3. d even, n even, and $n>d$. Choose $s=1$ and $r=\tfrac{1}{2}(n-d)$. Then r and s are natural numbers and $r+(r+d)=n$. In this case, there are just two terms in the arithmetic progression.

Case 4. d even, n odd, and $n>\tfrac{1}{2}dp(p-1)$. The argument is the same as for Case 2, and n is the sum of p terms in arithmetic progression.

This completes the proof of the general result. Finally, we consider what it looks like in the cases $d=1$, 2, and 3.

Corollary 1: A natural number > 1 is a sum of (at least two) consecutive natural numbers if and only if it is not a power of 2.

Proof: From the Theorem (with $d = 1$), if n is a sum of consecutive natural numbers, then it cannot be a power of 2.

Conversely, suppose n is not a power of 2. We can write $n = 2^h p^a q$, where $q \geq 1$ and the prime factors of q (if any) are all greater than p. If $q > 1$, then $q > p$ and $n > p^2 > \frac{1}{2} p(p-1)$ and, by the Theorem, n is expressible in the manner required. If $q = 1$ and $a > 1$, then $n \geq p^2 > \frac{1}{2} p(p-1)$ and, again, n is so expressible. If $q = 1$ and $a = 1$, then $n = 2^h p$. We need either $p > 2^{h+1} - 1$ or $2^h p > \frac{1}{2} p(p-1)$, i.e., either $p > 2^{h+1} - 1$ or $p < 2^{h+1} + 1$. One of these must hold, so that n is expressible as a sum of consecutive natural numbers.

Corollary 2: A natural number > 1 is a sum of natural numbers which form an arithmetic progression with common difference 2 if and only if it is not prime.

Proof: It is easy to see that, if n is prime, then it does not satisfy the conditions (2) in the Theorem with $d = 2$, so that n is not expressible in the way required. Suppose n is not prime. If n is even, it is greater than 2. If n is odd, say $n = pq$, for some odd integer $q \geq p$, then $n \geq p^2 > p(p-1)$. Hence, n is expressible in the manner required.

Corollary 3: A natural number > 1 is a sum of natural numbers which form an arithmetic progression with common difference 3 if and only if it is not one of the following:

(a) a power of 2;

(b) $2^h p$, where p is an odd prime such that $\frac{1}{3}(2^{h+1} + 1) < p \leq 3(2^{h+1} - 1)$.

Proof: If n is a power of 2, then, from the Theorem (with $d = 3$), it is not expressible in the way required. If $n = 2^h p$, where p is an odd prime, then n is expressible in the way required if and only if either $p > 3(2^{h+1} - 1)$ or $2^h p > \frac{3}{2} p(p-1)$. The latter is equivalent to $p \leq \frac{1}{3}(2^{h+1} + 1)$. Hence, n is not expressible in this way if and only if p lies between these two values, viz:

$$\frac{1}{3}(2^{h+1} + 1) < p \leq 3(2^{h+1} - 1).$$

If $n = 2^h pq$, where p is an odd prime and q is an odd number such that $q \geq p$, then, if $h = 0$ and $m = pq > 3(2^1 - 1)$, and if $h > 0$, then $n = 2^h pq \geq 2p^2 > \frac{3}{2} p(p-1)$, so that n is expressible in the way required.

Thus, examples of natural numbers *not* expressible as a sum of natural numbers in arithmetic progression with common difference 3 are:

$$\begin{aligned}
h = 0&: \quad 2^0 \times 3, \\
h = 1&: \quad 2 \times 3, 2 \times 5, 2 \times 7, \\
h = 2&: \quad 2^2 p, p \text{ prime}, \ 5 \leq p \leq 19, \\
h = 3&: \quad 2^3 p, p \text{ prime}, \ 7 \leq p \leq 43.
\end{aligned}$$

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