ON TRIANGULAR RECTANGULAR NUMBERS

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1. INTRODUCTION

A positive integer n is a **triangular** number if there is another positive integer k such that $n = \frac{1}{2}k(k+1)$. n is a **square** number if there is a positive integer ℓ such that $n = \ell^2$, and n is a **nearly square** number if there is a positive integer ℓ such that $n = \ell(\ell+1)$ (see [1], [4]). More generally, let σ be any nonnegative integer; a positive integer n will be called a **σ-rectangular** number if there is a positive integer ℓ such that $n = \ell(\ell+\sigma)$. Using this definition, a square number is a 0-rectangular number, and a nearly square number is a 1-rectangular number. It is not difficult to show that any positive integer n is an $(n-1)$-rectangular number, and an integer can be a σ-rectangular number for two different values of σ. We consider here the following problem: for a given nonnegative integer σ, generate all the triangular σ-rectangular numbers.

2. A PELLIAN EQUATION

Let n be a triangular σ-rectangular number, then

$$n = \frac{1}{2}k(k+1) = \ell(\ell+\sigma). \tag{1}$$

Since $8n+1 = (2k+1)^2$ and $4n+\sigma^2 = (2\ell+\sigma)^2$, it follows that $r^2 - 2s^2 = 1 - 2\sigma^2$ for $r = 2k+1$ and $s = 2\ell+\sigma$. Hence, we have the following result.

Theorem 1 Let $\sigma \geq 0$, $r \geq 1$, and $s \geq 0$ be three integers such that

$$r^2 - 2s^2 = 1 - 2\sigma^2, \tag{2}$$

and let

$$\begin{bmatrix} k \\ \ell \end{bmatrix} = \frac{1}{2}\begin{bmatrix} r-1 \\ s-\sigma \end{bmatrix}. \tag{3}$$

Then $\frac{1}{2}k(k+1) = \ell(\ell+\sigma)$. Furthermore, any triangular σ-rectangular number can be obtained in this way. □

By direct substitution, if (r, s) is any solution of (2), and we let

$$\begin{bmatrix} r' \\ s' \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} r \\ s \end{bmatrix} \quad \left(\text{or } \begin{bmatrix} r' \\ s' \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} r \\ s \end{bmatrix}\right), \tag{4}$$

then (r', s') is also a solution of (2). It follows directly that if (k, ℓ) are such that $\frac{1}{2}k(k+1) = \ell(\ell+\sigma)$, and if we let

$$\begin{bmatrix} k' \\ \ell' \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} k \\ \ell \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ \sigma \end{bmatrix}, \tag{5}$$

then $\frac{1}{2}k'(k'+1) = \ell'(\ell'+\sigma)$.

3. THE CASE $\sigma = 0$: TRIANGULAR SQUARE NUMBERS

It is known that the "smallest" solution (or the fundamental solution) of (2) for $\sigma = 0$ is $r_0 = 1$ and $s_0 = 0$ (see [2], [3], [5]). Furthermore, all the solutions of (2) are generated by the following recursive scheme: $r_0 = 1$, $s_0 = 0$, and

$$\begin{bmatrix} r_{i+1} \\ s_{i+1} \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} r_i \\ s_i \end{bmatrix} \quad (i = 0, 1, 2, \ldots). \tag{6}$$

Hence, any triangular square number can be obtained from the recursive scheme: $k_0 = 0$, $\ell_0 = 0$, and

$$\begin{bmatrix} k_{i+1} \\ \ell_{i+1} \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} k_i \\ \ell_i \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad (i = 0, 1, 2, \ldots). \tag{7}$$

TABLE 1. Triangular Square Numbers

	$n_i = \frac{1}{2}k_i(k_i+1) = \ell_i^2$		
i	k_i	ℓ_i	n_i
0	0	0	0
1	1	1	1
2	8	6	36
3	49	35	1225
4	288	204	41616
5	1681	1189	1413721

4. THE CASE $\sigma > 0$

Let us observe that $r = 1$ and $s = \sigma$ is always a solution of (2). With this initial value we can generate infinitely many solutions of (2) using (6). But it happens that this sequence of solutions does not contain all the solutions of (2) for some values of σ. We are led to the problem of finding all the "smallest" (or fundamental) solutions of (2). This problem is addressed elsewhere for more general pellian equations ([2], [3], [5]). We present here a simple proof for equation (2) using Fermat's descent method ([3], [5]). The method is based on the next two lemmas.

Lemma 2: Let $\sigma > 0$. If (r, s) is any solution of (2), then r is odd, $|s| \geq \sigma$, and $|s| >$ (=, <, resp.) $\sqrt{2\sigma^2 - 1}$ if any only if $|r| >$ (=, <, resp.) $|s|$.

Proof: Equation (2) is equivalent to $2(s^2 - \sigma^2) = r^2 - 1$ and $r^2 - s^2 = s^2 - (2\sigma^2 - 1)$. □

Lemma 3: Let $\sigma > 0$. Assume that (r, s) and $(\tilde{r}, \tilde{s})$ are two solutions of (2) such that

$$\begin{bmatrix} \tilde{r} \\ \tilde{s} \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix}.$$

(a) If $r \geq 0$ and $s > \sqrt{2\sigma^2 - 1}$, then $0 < \tilde{s} < s$, $-\tilde{s} < \tilde{r} < r$, and it follows that if

$$\begin{aligned} \tilde{s} &> \sqrt{2\sigma^2-1} &\text{then}\quad & \tilde{r} > \tilde{s}, \\ \tilde{s} &= \sqrt{2\sigma^2-1} &\text{then}\quad & \tilde{r} = \tilde{s}, \\ \tilde{s} &< \sqrt{2\sigma^2-1} &\text{then}\quad & |\tilde{r}| < \tilde{s}. \end{aligned}$$

(b) If $|r| < s$ and $0 < s < \sqrt{2\sigma^2 - 1}$, then $\tilde{s} > s$, $|\tilde{r}| > \tilde{s}$, and it follows that $\tilde{s} > \sqrt{2\sigma^2 - 1}$.

Proof: ***(a)*** From Lemma 2, $r > s > \sqrt{2\sigma^2 - 1}$. Then $\tilde{s} = s - 2(r - s) < s$ and $\tilde{s} = r - 3(r - s) = (s(r-s) + 2(2\sigma^2 - 1)) / (r + s) > 0$ because $(r - s)(r + s) = s^2 - (2\sigma^2 - 1)$. Also, if $\tilde{r} + \tilde{s} = r - s > 0$, then $\tilde{r} > -\tilde{s}$ and $\tilde{r} < r$. We complete the proof using Lemma 2.

(b) Since $\tilde{s} = s + 2(s - r)$ and $\tilde{r} = -\tilde{s} + (r - s)$, we have $\tilde{s} > s > 0$ and $\tilde{r} < -\tilde{s}$. Then $|\tilde{r}| > \tilde{s}$ and hence $\tilde{s} > \sqrt{2\sigma^2 - 1}$. □

Definition 4: Let $\sigma > 0$. A fundamental solution for (2) is a solution (r, s) of (2) such that

$$\sigma \leq s \leq \sqrt{2\sigma^2 - 1} \quad \text{and} \quad -s < r \leq s. \; \square$$

Finally, using Fermat's descent method, we have the following result.

Theorem 5: Let $\sigma > 0$. For any positive solution (r, s) of (2), there exists a unique fundamental solution (r_0, s_0) of (2) and a nonnegative integer i such that

$$\begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}^i \begin{bmatrix} r_0 \\ s_0 \end{bmatrix}. \; \square$$

To find all the fundamental solutions of (2) for a given σ, we can consider a systematic method based on the following facts:

(i) $(1, \sigma)$ is always a fundamental solution,
(ii) r is always odd,
(iii) s and σ have the same parity.

Hence, for a given σ we can consider s with the parity of σ in the interval $[\sigma, \sqrt{2\sigma^2 - 1}]$ for which $r = \sqrt{1 - 2\sigma^2 + 2s^2}$ is an integer. Table 2 presents the fundamental solutions of (2) for σ = 1, ..., 30. Let us remark that if $2\sigma^2 - 1$ is a prime number, (2) has no fundamental solution but $(\pm 1, \sigma)$ (see [2], Theorem 110).

Finally, to generate the triangular σ-rectangular numbers, we consider the fundamental solutions (r_0, s_0) of (2) and

(i) if $r_0 > 0$, then $\begin{bmatrix} k_0 \\ \ell_0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} r_0 - 1 \\ s_0 - \sigma \end{bmatrix}$,

(ii) if $r_0 < 0$, then $\begin{bmatrix} k_0 \\ \ell_0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} r_0 \\ s_0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ \sigma \end{bmatrix}$, and we use (5). We associate a class of triangular σ-rectangular numbers to each fundamental solution of (2), and the classes are distinct.

TABLE 2. Fundamental Solutions of (2) for $\sigma = 1, \ldots, 30$

σ	$2\sigma^2-1$	(r, s)	σ	$2\sigma^2-1$	(r, s)
1	1*	(1, 1)	16	511	(±1, 16), (±17, 20)
2	7•	(±1, 2)	17	577•	(±1, 17)
3	17•	(±1, 3)	18	647•	(±1, 18)
4	31•	(±1, 4)	19	721	(±1, 19), (±23, 25)
5	49*	(±1, 5), (7, 7)	20	799	(±1, 20), (±13, 22)
6	71•	(±1, 6)	21	881•	(±1, 21)
7	97•	(±1, 7)	22	967•	(±1, 22)
8	127•	(±1, 8)	23	1057	(±1, 23), (±25, 29)
9	161	(±1, 9), (±9, 11)	24	1151•	(±1, 24)
10	199•	(±1, 10)	25	1249•	(±1, 25)
11	241•	(±1, 11)	26	1351	(±1, 26), (±31, 34)
12	287	(±1, 12), (±15, 16)	27	1457	(±1, 27), (±15, 29)
13	337•	(±1, 13)	28	1567•	(±1. 28)
14	391	(±1, 14), (±11, 16)	29	1681*	(±1, 29), (±41, 41)
15	449•	(±1, 15)	30	1799	(±1, 30), (±33, 38)

* a square number; • a prime number

Example 6: Consider $\sigma = 12$. Using (5), we have

$$\begin{bmatrix} k_{i+1} \\ \ell_{i+1} \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} k_i \\ \ell_i \end{bmatrix} + \begin{bmatrix} 25 \\ 13 \end{bmatrix} \quad (i = 0, 1, 2, \ldots),$$

where the (k_0, ℓ_0) are as given in Table 3. In this case, there exist four different classes of triangular 12-rectangular numbers. □

TABLE 3. Initial Values (k_0, ℓ_0)

$\sigma = 12$		$\sigma = 29$	
(r_0, s_0)	(k_0, ℓ_0)	(r_0, s_0)	(k_0, ℓ_0)
(1, 12)	(0, 0)	(1, 29)	(0, 0)
(−1, 12)	(22, 11)	(−1, 29)	(56, 28)
(15, 16)	(7, 2)	(41, 41)	(20, 6)
(−15, 16)	(9, 3)		

Example 7: Consider $\sigma = 29$. Using (5), we have

$$\begin{bmatrix} k_{i+1} \\ \ell_{i+1} \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} k_i \\ \ell_i \end{bmatrix} + \begin{bmatrix} 59 \\ 30 \end{bmatrix} \quad (i = 0, 1, 2, \ldots),$$

where the (k_0, ℓ_0) are as given in Table 3 above. In this case, there exist three different classes of triangular 29-rectangular numbers. □

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