CONJECTURES CONCERNING IRRATIONAL NUMBERS AND INTEGERS

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Let r be an irrational number between one and two. Every positive integer n can be represented in terms of r in a very simple way (Theorem 1) that perhaps deserves to be better known than it is. To get started, recall the customary notation [7] associated with the continued fraction for r:

$$r=[a_0,a_1,a_2,\dots], \tag{1}$$

$$p_{-2}=0,\ p_{-1}=1,\ p_i=a_ip_{i-1}+p_{i-2}$$

and

$$q_{-2}=1,\ q_{-1}=0,\ q_i=a_iq_{i-1}+q_{i-2},$$

for $i=0,1,2,\dots$. The rational numbers p_i/q_i are in reduced form, and their limit is r. Moreover,

$$1=q_0\le q_1<q_2<\dots<q_i<\cdots. \tag{2}$$

Theorem 1: Every positive integer n has a representation

$$n=\sum_{i=0}^{u}c_iq_i, \tag{3}$$

where the c_i are integers satisfying

$$0\le c_i\le a_{i+1} \text{ for } 0\le i\le u, \text{ and } c_u\ge 1. \tag{4}$$

Proof: For given n, let u be the index for which $q_u\le n<q_{u+1}$. By the division algorithm, there exist integers c_u and n_{u-1} such that $n=c_uq_u+n_{u-1}$, where $0\le n_{u-1}<q_u$. Now

$$(a_{u+1}+1)q_u\ge a_{u+1}q_u+q_{u-1}=q_{u+1}>n,$$

so that $c_u\le a_{u+1}$. If $n_{u-1}>0$ then, similarly, $n_{u-1}=c_{u-1}q_{u-1}+n_{u-2}$, where $0\le n_{u-2}<q_{u-1}$ and $c_{u-1}\le a_u$, so that $n=c_uq_u+c_{u-1}q_{u-1}+n_{u-2}$. If $n_{u-2}>0$, we continue to strip away terms of the form c_iq_i until reaching the representation (3). □

The proof of Theorem 1 occurs within a proof of a deeper theorem [3, p. 125] which is not primarily concerned with representing integers. (Theorem 1 may be viewed as a corollary to a more general representation theorem; see [1], [8, Ch. 8], and [4].) We abbreviate the representation (3) as $CF(r,n)$ and the set of all such representations for given r as $CF(r,\cdot)$. By construction, $CF(r,\cdot)$ is a unique representation in the sense that the coefficients c_i are the only positive integers satisfying

$$0\le n-\sum_{i=s}^{u}c_iq_i<q_s \tag{4}$$

for $s=0,1,\dots,u$.

Note that in (2) the base numbers are distinct except perhaps for $q_1 = q_0$. We shall show that when this happens either $c_0 = 0$ or else $c_1 = 0$; that is, the base number 1 occurs at most once in each evaluation of (3). For a proof, suppose that the proposition is false for some r, and let n be the least positive integer having $CF(r, n)$ of the form

$$n = c_0 \cdot 1 + c_1 \cdot 1 + c_2 \cdot q_2 + \cdots + c_u \cdot q_u$$

with c_0 and c_1 both nonzero. Let $n' = n - c_2 q_2 - \cdots - c_u q_u$. If $c_1 \le a_2 - 1$, then $1 \cdot 1 + c_1 \cdot 1$ and $0 \cdot 1 + (c_1 + 1) \cdot 1$ are distinct representations of n', contrary to the uniqueness of $CF(r, n')$. On the other hand, if $c_1 = a_2$, then $c_0 = 1$ since $c_0 \le a_1 = 1$, so that $c_0 + c_1 = a_2 + 1$. However, $a_2 + 1 = q_2$, so that $1 \cdot 1 + a_2 \cdot q_1 = 0 \cdot q_0 + 0 \cdot q_1 + 1 \cdot q_2$, contrary to the uniqueness of $CF(r, q_2)$.

Let s_j $[= s_j(r)]$ be the j^{th} positive integer n for which $c_1 \ne 0$ in the representation $CF(r, n)$. That is, s_j is the j^{th} positive integer n for which the smallest base number appearing in (3) is 1. Our first conjecture is that the sequence $\{s_j\}$ is "almost" an arithmetic sequence.

Conjecture 1: There exists a number $f = f(r)$ such that $|s_j - jf| \le 2$ for all $j \ge 1$.

In order to state a second conjecture about the sequence $\{s_j\}$, we recall a definition introduced by I. Niven [6]. Suppose $\Lambda = \{\lambda_j\}$ is a sequence of integers. For any integers k and $m \ge 2$, let $\Lambda(J, k, m)$ be the number of indices j that satisfy $1 \le j < J$ and $\lambda_j \equiv k \pmod{m}$. If the limit

$$\lim_{J \to \infty} \frac{1}{J} \Lambda(J, k, m)$$

exists and equals $1/m$ for all k satisfying $1 \le k \le m$, then Λ is *uniformly distributed* (*mod m*). If Λ is uniformly distributed (mod m) for every integer $m \ge 2$, then Λ is *uniformly distributed.*

Conjecture 2: $\{s_j\}$ is uniformly distributed.

Conjectures 1 and 2 extend to other sequences. Let $s(i, j)$ be the j^{th} positive integer n for which the least base number appearing in (3) is q_j.

Conjecture 3: There exist numbers $f_i = f_i(r)$ and $B_i = B_i(r)$ such that $|s(i, j) - jf_i| \le B_i$ for all $j \ge 1$.

Conjecture 4: For each $i \ge 1$, the sequence $\{s(i, j)\}_{j=1}^{\infty}$ is uniformly distributed.

The simplest representations $CF(r, \cdot)$ are for $r = (1 + \sqrt{5})/2$, for in this case $a_i = 1$ for all $i \ge 0$, so that (3) gives the well-studied Zeckendorf representation of n. Moreover, the array $\{s(i, j)\}$ is the Zeckendorf array, which is proved identical in [2] to the Wythoff array introduced in [5]. For general r, we suggest that $CF(r, \cdot)$ be called the *r-Zeckendorf representation of n* and that the array $\{s(i, j)\}$ be called the *r-Zeckendorf array*.

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