

CONJECTURES CONCERNING IRRATIONAL NUMBERS AND INTEGERS

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Let r be an irrational number between one and two. Every positive integer n can be represented in terms of r in a very simple way (Theorem 1) that perhaps deserves to be better known than it is. To get started, recall the customary notation [7] associated with the continued fraction for r :

$$r = [a_0, a_1, a_2, \dots], \quad (1)$$

and

$$p_{-2} = 0, \quad p_{-1} = 1, \quad p_i = a_i p_{i-1} + p_{i-2}$$

$$q_{-2} = 1, \quad q_{-1} = 0, \quad q_i = a_i q_{i-1} + q_{i-2},$$

for $i = 0, 1, 2, \dots$. The rational numbers p_i / q_i are in reduced form, and their limit is r . Moreover,

$$1 = q_0 \leq q_1 < q_2 < \dots < q_i < \dots \quad (2)$$

Theorem 1: Every positive integer n has a representation

$$n = \sum_{i=0}^u c_i q_i, \quad (3)$$

where the c_i are integers satisfying

$$0 \leq c_i \leq a_{i+1} \text{ for } 0 \leq i \leq u, \text{ and } c_u \geq 1. \quad (4)$$

Proof: For given n , let u be the index for which $q_u \leq n < q_{u+1}$. By the division algorithm, there exist integers c_u and n_{u-1} such that $n = c_u q_u + n_{u-1}$, where $0 \leq n_{u-1} < q_u$. Now

$$(a_{u+1} + 1)q_u \geq a_{u+1}q_u + q_{u-1} = q_{u+1} > n,$$

so that $c_u \leq a_{u+1}$. If $n_{u-1} > 0$ then, similarly, $n_{u-1} = c_{u-1}q_{u-1} + n_{u-2}$, where $0 \leq n_{u-2} < q_{u-1}$ and $c_{u-1} \leq a_u$, so that $n = c_u q_u + c_{u-1} q_{u-1} + n_{u-2}$. If $n_{u-2} > 0$, we continue to strip away terms of the form $c_i q_i$ until reaching the representation (3). \square

The proof of Theorem 1 occurs within a proof of a deeper theorem [3, p. 125] which is not primarily concerned with representing integers. (Theorem 1 may be viewed as a corollary to a more general representation theorem; see [1], [8, Ch. 8], and [4].) We abbreviate the representation (3) as $CF(r, n)$ and the set of all such representations for given r as $CF(r, \cdot)$. By construction, $CF(r, \cdot)$ is a unique representation in the sense that the coefficients c_i are the only positive integers satisfying

$$0 \leq n - \sum_{i=s}^u c_i q_i < q_s \quad (4)$$

for $s = 0, 1, \dots, u$.

Note that in (2) the base numbers are distinct except perhaps for $q_1 = q_0$. We shall show that when this happens either $c_0 = 0$ or else $c_1 = 0$; that is, the base number 1 occurs at most once in each evaluation of (3). For a proof, suppose that the proposition is false for some r , and let n be the least positive integer having $CF(r, n)$ of the form

$$n = c_0 \cdot 1 + c_1 \cdot 1 + c_2 \cdot q_2 + \cdots + c_u \cdot q_u$$

with c_0 and c_1 both nonzero. Let $n' = n - c_2 q_2 - \cdots - c_u q_u$. If $c_1 \leq a_2 - 1$, then $1 \cdot 1 + c_1 \cdot 1$ and $0 \cdot 1 + (c_1 + 1) \cdot 1$ are distinct representations of n' , contrary to the uniqueness of $CF(r, n')$. On the other hand, if $c_1 = a_2$, then $c_0 = 1$ since $c_0 \leq a_1 = 1$, so that $c_0 + c_1 = a_2 + 1$. However, $a_2 + 1 = q_2$, so that $1 \cdot 1 + a_2 \cdot q_1 = 0 \cdot q_0 + 0 \cdot q_1 + 1 \cdot q_2$, contrary to the uniqueness of $CF(r, q_2)$.

Let $s_j [= s_j(r)]$ be the j^{th} positive integer n for which $c_1 \neq 0$ in the representation $CF(r, n)$. That is, s_j is the j^{th} positive integer n for which the smallest base number appearing in (3) is 1. Our first conjecture is that the sequence $\{s_j\}$ is "almost" an arithmetic sequence.

Conjecture 1: There exists a number $f = f(r)$ such that $|s_j - jf| \leq 2$ for all $j \geq 1$.

In order to state a second conjecture about the sequence $\{s_j\}$, we recall a definition introduced by I. Niven [6]. Suppose $\Lambda = \{\lambda_j\}$ is a sequence of integers. For any integers k and $m \geq 2$, let $\Lambda(J, k, m)$ be the number of indices j that satisfy $1 \leq j < J$ and $\lambda_j \equiv k \pmod{m}$. If the limit

$$\lim_{J \rightarrow \infty} \frac{1}{J} \Lambda(J, k, m)$$

exists and equals $1/m$ for all k satisfying $1 \leq k \leq m$, then Λ is *uniformly distributed (mod m)*. If Λ is uniformly distributed (mod m) for every integer $m \geq 2$, then Λ is *uniformly distributed*.

Conjecture 2: $\{s_j\}$ is uniformly distributed.

Conjectures 1 and 2 extend to other sequences. Let $s(i, j)$ be the j^{th} positive integer n for which the least base number appearing in (3) is q_j .

Conjecture 3: There exist numbers $f_i = f_i(r)$ and $B_i = B_i(r)$ such that $|s(i, j) - jf_i| \leq B_i$ for all $j \geq 1$.

Conjecture 4: For each $i \geq 1$, the sequence $\{s(i, j)\}_{j=1}^{\infty}$ is uniformly distributed.

The simplest representations $CF(r, \cdot)$ are for $r = (1 + \sqrt{5})/2$, for in this case $a_i = 1$ for all $i \geq 0$, so that (3) gives the well-studied Zeckendorf representation of n . Moreover, the array $\{s(i, j)\}$ is the Zeckendorf array, which is proved identical in [2] to the Wythoff array introduced in [5]. For general r , we suggest that $CF(r, \cdot)$ be called the *r-Zeckendorf representation of n* and that the array $\{s(i, j)\}$ be called the *r-Zeckendorf array*.

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