CONCERNING THE RECURSIVE SEQUENCE

\[ A_{n+k} = \sum_{i=1}^{k} a_i A_{n+i-1}^{a_i} \]

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1. MAIN RESULT

In [1] H. T. Freitag has raised a conjecture that for the sequence \( \{A_n\} \), defined by \( A_{n+2} = \sqrt{A_{n+1} + A_n} \) for all \( n \geq 1 \), \( \lim_{n \to \infty} A_n = 4 \) regardless of the choice of \( A_1, A_2 > 0 \). In this note we will give a positive answer to this conjecture by proving the following more general theorem.

**Theorem 1:** If \(-1 < \alpha_i < 1, 1 \leq i \leq k \) and \( A_{n+k} = \sum_{i=1}^{k} a_i A_{n+i-1}^{a_i}, n \geq 1 \), then

\[ \lim_{n \to \infty} A_n = L, \]

the unique root of the equation \( \sum_{i=1}^{k} a_i x^{a_i} = 1 \) in the interval \((0, \infty)\), regardless of the choice of \( A_i > 0, 1 \leq i \leq k \). where \( a_i \geq 0, 1 \leq i \leq k \), and \( \sum_{i=1}^{k} a_i > 0. \)

In particular, if \( k = 2, a_1 = a_2 = 1 \), and \( a_1 = a_2 = \frac{1}{2} \), we have

\[ \lim_{n \to \infty} A_n = 4. \]

This coincides with Freitag's conjecture.

**Proof:** Let \( A_n = Lx_n \). Then

\[ x_{n+k} = \sum_{i=1}^{k} \beta_i x_{n+i-1}^{a_i}, \]

where \( \beta_i = a_i L^{a_i-1} \), and therefore

\[ \sum_{i=1}^{k} \beta_i = 1. \quad (1) \]

Obviously, we only need to prove that

\[ \lim_{n \to \infty} x_n = 1. \quad (2) \]

To this end, set \( M = \max\{x_i, x_i^{-1} \} \) and \( \alpha = \max\{|\alpha_1|, \ldots, |\alpha_k|\} \). It is obvious that \( M \geq 1, 0 \leq \alpha < 1 \), and

\[ M \geq x_i \geq M^{-\alpha}, 1 \leq i \leq k. \quad (3) \]

We will use induction to prove that

\[ M^{\alpha^i} \geq x_{n+i} \geq M^{-\alpha^i}, 1 \leq i \leq k. \quad (4) \]

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CONCERNING THE RECURSIVE SEQUENCE $A_{n+k} = \sum_{i=1}^{k} a_i A_{n+i-1}$

holds for all $n \geq 0$. In fact, from (3), (4) holds when $n = 0$. We assume that (4) holds if $n \leq \ell - 1$. For $n = \ell$, from the induction assumption and the definition of $M$, it follows that

$$M^{\alpha i} \geq M^{\alpha i + 1}, \quad 1 \leq i \leq k,$$

(5)

and

$$M^{-\alpha i} \leq x_{(\ell-1)k+i} \leq M^{\alpha i + 1}, \quad 1 \leq i \leq k.$$ 

(6)

Therefore, from (5) and (6), we have

$$x_{k \ell+1} = \sum_{i=1}^{k} \beta_i x_{(\ell-1)k+i+1} \leq \sum_{i=1}^{k} \beta_i M^{\alpha i + 1} \leq M^{\alpha \ell},$$

and, furthermore, we have

$$x_{k \ell+2} = \sum_{i=1}^{k} \beta_i x_{(\ell-1)k+i+1} \leq \sum_{i=1}^{k-1} \beta_i M^{\alpha i + 1} + \beta_k M^{\alpha k} \leq M^{\alpha \ell}.$$ 

In the last step we have used the fact that $M^{\alpha k} \leq M^{\alpha \ell}$. Similarly, the left-hand inequality of (4) holds for $n = \ell$ and other indices $i$, $3 \leq i \leq k$. The right-hand inequality of (4) can be justified in a similar way. Noting that $0 \leq \alpha < 1$, we obtain

$$\lim_{n \to \infty} M^{-\alpha n} = \lim_{n \to \infty} M^{\alpha n} = 1.$$ 

By (4), this implies that (2) holds.

\textbf{Corollary 1:} If $-1 < \alpha_1 = \ldots = \alpha_k = \alpha < 1$ and $a_1 = \ldots a_k = 1$, then

$$\lim_{n \to \infty} A_n = k^{1/\alpha},$$

independent of the choice of $A_1, A_2, \ldots, A_k > 0$, where $\{A_n\}_1^{\infty}$ is as defined in Theorem 1.

\textbf{Corollary 2:} If $-1 < \alpha_i < 1$, $a_i \geq 0$, and $\sum_{i=1}^{k} a_i = 1$, then

$$\lim_{n \to \infty} A_n = 1,$$

independent of the choice of $A_1, A_2, \ldots, A_k > 0$, where $\{A_n\}_1^{\infty}$ is also as defined in Theorem 1. Corollary 2 follows from the fact that $L = 1$ is the only root of the equation $\sum_{i=1}^{k} a_i x^{\alpha_i - 1} - 1 = 0$ in the interval $(0, \infty)$.

\section{2. FURTHER RESULTS}

In this section we consider a \textit{linear} recursive sequence, that is, when we choose $\alpha_i = 1$, $1 \leq i \leq k$, in the recursive sequence considered above.

\textbf{Theorem 2:} Let the complex sequence $\{A_n\}_1^{\infty}$ satisfy

$$A_{n+k} = \sum_{i=1}^{k} a_i A_{n+i-1}.$$ 

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CONCERNING THE RECURSIVE SEQUENCE $A_{n+k} = \sum_{i=1}^{k} a_i A_{n+i-1}^i$

Then, if $a_i > 0$, $1 \leq i \leq k$, and $\sum_{i=1}^{k} a_i = 1$, the sequence $\{A_n\}_{1}^{\infty}$ converges to a limit which depends on the values of $A_i$, $1 \leq i \leq k$.

**Proof:** We will prove that $x = 1$ is a single root of the eigenpolynomial,

$$p(x) = x^k - \sum_{i=1}^{k} a_i x^{i-1} = 0,$$

(7)

of the recursive sequence

$$A_{n+k} = \sum_{i=1}^{k} a_i A_{n+i-1}^i,$$

and the moduli of all other roots of (7) are less than 1.

In fact, since $\sum_{i=1}^{k} a_i = 1$, we have $p(1) = 0$. This means that $x = 1$ is a root of $p(x)$. From

$$p'(1) = k - \sum_{i=1}^{k} (i-1)a_i \geq 1,$$

it follows that $x = 1$ is a single root of $p(x)$. On the other hand, for $x = re^{i\theta}$, $r \geq 1$, and $0 \leq \theta < 2\pi$, we have

$$|p(re^{i\theta})| \geq r^k - \left| \sum_{j=1}^{k} a_j r^{j-1} e^{(j-1)i\theta} \right| \geq \left( r - \sum_{j=1}^{k} a_j \right) r^{k-1} \geq 0.$$

It is easy to see that the above inequalities become equalities if and only if $r = 1$ and $\theta = 0$. Therefore, if $x = x_0$ is a zero of $p(x)$, then $|x_0| < 1$ and $x_0 = 1$ when $|x_0| = 1$. Set

$$p(x) = (x-1)(x-x_1)^{r_1} \cdots (x-x_m)^{r_m},$$

(8)

where $1 + r_1 + \cdots + r_m = k$, $|x_j| < 1$, $1 \leq j \leq m$, and $x_i \neq x_j$ when $i \neq j$. It is well known that $\{A_n\}_{1}^{\infty}$ has the general solution

$$A_n = c + \sum_{j=0}^{m} \sum_{i=1}^{r_i} c_{i,j} n^{i-1} x_j^n.$$

(9)

From (9), we deduce that

$$\lim_{n \to \infty} A_n = c.$$

The value of $c$ depends on the choice of $A_j$, $1 \leq j \leq k$. This completes the proof of Theorem 2. □

**Note:** Theorem 1 and Theorem 2 can be generalized easily to discuss sequences of functions. To state this precisely, we have

**Theorem 3:** Let $a_i = a_i(x)$ and $\alpha_i = \alpha_i(x)$, $1 \leq i \leq k$, be functions defined on a point set $I \subset \mathbb{R}^m$, a Euclidean space, and let the function sequence $\{A_n(x)\}_{1}^{\infty}$ be defined as

$$A_{n+k}(x) = \sum_{i=1}^{k} a_i A_{n+i-1}^i(x), \quad n \geq 1.$$
CONCERNING THE RECURSIVE SEQUENCE \( A_{n+k} = \sum_{i=1}^{k} a_i A^{\beta}_{n+i-1} \)

Then we have:

(1) If \( a_i(x) \geq 0 \) and \(-1 < a_i(x) < 1\) hold for an \( x \in I \), \( \{A_n(x)\}_1^\infty \) converges at the point \( x \) to \( L = L(x) \), the unique root of \( \sum_{i=1}^{k} a_i \alpha_i^{\beta_i-1} = 1 \) if \( a_i(x) \), \( 1 \leq i \leq k \), are not all zeros and the sequence converges pointwise to zero if \( a_i(x) = 0 \) for all \( i, 1 \leq i \leq k \), regardless of the choice of \( A_i(x) > 0, 1 \leq i \leq k \);

(2) If \( a_i(x) \geq 0, \sum_{i=1}^{k} a_i(x) = 1, \) and \( a_i(x) = 1, 1 \leq i \leq k \), hold for an \( x \in I \), \( \{A_n(x)\}_1^\infty \) converges at the point \( x \). In particular, for case (1), \( \{A_n(x)\}_1^\infty \) converges uniformly if there are constants \( \alpha, 0 \leq \alpha < 1, \ a > 0, \) and \( M \) such that \( |a_i(x)| \leq \alpha, 1 \leq i \leq k, \ 0 < \sum_{i=1}^{k} a_i(x) \leq a, \ x \in I, \) and \( \sup_{x \in I} \{A_i(x), A_i^{-1}(x) | 1 \leq i \leq k \} \leq M \) hold, respectively.

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REFERENCE


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