A DISJOINT COVERING OF THE SET OF NATURAL NUMBERS CONSISTING OF SEQUENCES DEFINED BY A RECURRENCE WHOSE CHARACTERISTIC EQUATION HAS A PISOT NUMBER ROOT

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(*Submitted February 1994*)

1. INTRODUCTION

Burke and Bergum [1] called a family of sequences defined by a linear recurrence "a disjoint covering" if every natural number was contained in exactly one of the given sequences. They gave arithmetic progressions and geometric progressions as simple examples of finite and infinite disjoint coverings. Although they also constructed n^{th}-order recurrences that were disjoint coverings, the resulting sequences were not essentially n^{th} ordered since they were the same as the ones listed above and could be defined by first-order recurrences. Zöllner [4] proved that there is an infinite disjoint covering generated by the Fibonacci recurrence

$$u_{n+2} = u_{n+1} + u_n, \tag{1}$$

answering the question proposed in [1] affirmatively. This is the first paper establishing the existence of a disjoint covering consisting of sequences essentially defined by a second-order recurrence.

In this paper we will show the existence of disjoint coverings essentially generated by linear recurrences of any order.

2. A TYPE OF PISOT NUMBER

A Pisot number is a real algebraic integer greater than 1 such that the absolute value of every conjugate is less than 1 (see [2]). We consider a special type of Pisot number that satisfies a monic irreducible equation with integral coefficients

$$f(x) = x^m - a_1x^{m-1} - \cdots - a_{m-1}x - a_m = 0, \tag{2}$$

where $m \geq 2$, $a_1 > 0$, $a_i \geq 0$ (for $i = 2, \ldots, m-1$), and $a_m > 0$.

Since $f(1) < 0$, equation (2) has a real number solution $\alpha > 1$. Let β_i $(i = 1, 2, \ldots, m-1)$ be the other roots of (2).

Example: We will show that if

$$a_1 > 1 + a_2 + \cdots + a_m, \tag{3}$$

then $|\beta_i| < 1$ for $i = 1, 2, \ldots, m-1$, and α is a Pisot number.

Let $g(x) = x^m - a_1x^{m-1} - a_2x^{m-2} = x^{m-2}(x^2 - a_1x - a_2)$ and let $h(x) = a_3x^{m-3} + \cdots + a_{m-1}x + a_m$. Then we have

$$|g(x)| \geq a_1 - 1 - a_2 > a_3 + a_4 + \cdots + a_m \geq |h(x)|$$

for any complex value of x on the unit circle $|x| = 1$ by (3). Therefore, by Rouché's theorem (see [3]), the number of roots of equation (2), or $g(x) = h(x)$, in the unit circle is equal to one of $g(x) = 0$, which is $m-1$, since the equation $x^2 - a_1x - a_2 = 0$ has two real roots x_1 and x_2, where $x_1 > 1$, $-1 < x_2 < 0$.

Now we will show that $f(x)$ in (2) is irreducible when (3) is satisfied. If it is reducible, then it must be decomposed into monic polynomials with integral coefficients, and each factor must have at least one root of modulus greater than or equal to 1, since the product of its roots is an integer. This contradicts the fact we have just proved above. Thus, we have shown that there are equations of type (2) that have a Pisot number root for each $m \geq 2$.

Remark: It should be noticed that, if (2) has a Pisot number root, all roots of (2) are simple, since any irreducible polynomial with rational coefficients has no multiple root.

3. SEQUENCES DEFINED BY A LINEAR RECURRENCE

We consider the recurrence

$$u_n = a_1u_{n-1} + a_2u_{n-2} + \cdots + a_mu_{n-m} \tag{4}$$

that has $f(x)$ in (2) as its characteristic polynomial. Let S be the set of all the positive integer sequences $\{u_n\}$ defined by recurrence (4). In the following we will establish that the existence of a disjoint covering of the set of natural numbers consists of the sequences in S.

According to the Remark in the previous section, the general term of $\{u_n\}$ is expressed as

$$u_n = c_0\alpha^n + c_1\beta_1^n + \cdots + c_{m-1}\beta_{m-1}^n, \tag{5}$$

where $c_0 > 0$ because $\alpha^n \to \infty$ and $\beta_i^n \to 0$ for $i = 1, 2, \ldots, m-1$ as n tends to ∞.

Let us define m integer sequences $\{t_n^{(i)}\}$ (for $i = 1, 2, \ldots, m$) satisfying (4) with the initial conditions $t_j^{(i)} = \delta_{ij}$ $(i, j = 1, 2, \ldots, m)$, where the right-hand side is Kroneker's delta.

These sequences have zero or positive integer terms, and their general terms are expressed as

$$t_n^{(i)} = b_{i,0}\alpha^n + b_{i,1}\beta_1^n + \cdots + b_{i,m-1}\beta_{m-1}^n, \tag{6}$$

where $b_{i,0} > 0$ as c_0 in (5). The n^{th} term u_n of the sequence defined by recurrence (2) is expressed as a linear combination of leading m terms as

$$u_n = t_n^{(1)}u_1 + t_n^{(2)}u_2 + \cdots + t_n^{(m)}u_m, \tag{7}$$

where the coefficients consist of n^{th} terms of these sequences.

4. THE CONSTRUCTION OF A DISJOINT COVERING

Now we show that there exists a disjoint covering of the set of all natural numbers consisting of sequences in S following the method used in [4].

Notice that the set cannot be covered with a finite number of such sequences. In fact, as $c_0 > 0$ in (5), $u_n \sim c_0\alpha^n$ so that $u_{n+1} - u_n \to \infty$ as n tends to ∞.

First, we define the sequence $\{u_n^{(1)}\}$ in S with initial conditions $u_n^{(1)} = n$ (for $n = 1, 2, \ldots, m$). Then assuming that, for $i = 1, 2, \ldots, k-1$, we have determined mutually disjoint increasing sequences $\{u_n^{(i)}\}$ in S that satisfy the conditions

$$u_1^{(i)} = \min \overline{V}_{i-1}, \tag{8}$$

where $\overline{V}_i$ denotes the complement of the set $V_i = \{u_n^{(j)} \mid j = 1, 2, \ldots, i;\ n = 1, 2, 3, \ldots\}$ in the set of all natural numbers, and

$$u_n^{(i)} < u_n^{(j)} \text{ for } i < j, \tag{9}$$

we will show that we can choose the next sequence $\{u_n^{(k)}\}$ in S so that these k sequences are mutually disjoint and satisfy conditions (8) and (9).

We can see that $\overline{V}_i$ is always nonempty by using the statement made at the beginning of this section. Thus, we can put $u_1^{(k)} = \min \overline{V}_{k-1}$ and $u_r^{(k)} = \min \overline{V}_{k-1} - \{u_1^{(k)}, u_2^{(k)}, \ldots, u_{r-1}^{(k)}\}$ for $r = 2, 3, \ldots, m-1$.

Let $M_k = \max\{u_m^{(k-1)}, u_{m-1}^{(k)}\}$ and let L_k be an integer larger than M_k. If we take any integer in the interval $(M_k, L_k]$ as the value of $u_m^{(k)}$, the resulting sequence $\{u_n^{(k)}\}$ in S will satisfy the inequality $u_n^{(k-1)} < u_n^{(k)}$ for all n, but it will possibly have a common term with one of the sequences $\{u_n^{(i)}\}$ $(i = 1, 2, \ldots, k-1)$ already built. We will evaluate the number N of such integers to show that we can find an integer value of $u_m^{(k)}$ in the interval $(M_k, L_k]$ so that the resulting sequence $\{u_n^{(k)}\}$ does not overlap with any of the $k-1$ sequences already built if we take a large enough value for L_k.

Suppose that $u_p^{(k)} = u_q^{(i)}$ for some $i < k$. Then $m \le p < q$. Using the expression shown in (7) for $u_p^{(k)}$ and $u_q^{(i)}$, we have

$$t_p^{(1)} u_1^{(k)} + \cdots + t_p^{(m)} u_m^{(k)} = t_q^{(1)} u_1^{(i)} + \cdots + t_q^{(m)} u_m^{(i)}. \tag{10}$$

If $u_{p+r}^{(k)} = u_{q+r}^{(i)}$ for $r = 1, 2, \ldots, m-1$, then, using recurrence (4) to the opposite direction, we have $u_1^{(k)} = u_{q-p+1}^{(i)}$, which contradicts the choice of $u_1^{(k)}$. Thus, there is an r such that

$$0 \le r < m-1,\ u_{p+r}^{(k)} = u_{q+r}^{(i)}, \text{ and } u_{p+r+1}^{(k)} \ne u_{q+r+1}^{(i)}. \tag{11}$$

As we are going to find the largest p such that $u_p^{(k)}$ is equal to an element of V_{k-1}, replacing $p+r$ with p if necessary, we can assume that $r = 0$. Then, using expression (6), we have

$$u_{p+1}^{(k)} - \alpha u_p^{(k)} = -\sum\left\{b_{j,1}\beta_1^p(\alpha - \beta_1) + \cdots + b_{j,m-1}\beta_{m-1}^p(\alpha - \beta_{m-1})\right\} u_j^{(k)},$$

where summation runs from $j = 1$ to $j = m$. For $u_{q+1}^{(i)} - \alpha u_q^{(i)}$, we have a similar expression. Since we can see that

$$\left|\left(u_{p+1}^{(k)} - \alpha u_p^{(k)}\right) - \left(u_{q+1}^{(i)} - \alpha u_q^{(i)}\right)\right| = \left|u_{p+1}^{(k)} - u_{q+1}^{(i)}\right| \ge 1,$$

from (11), we have

$$\Big|\Sigma\{b_{j,1}\beta_1^p(\alpha-\beta_1)+\cdots+b_{j,m-1}\beta_{m-1}^p(\alpha-\beta_{m-1})\}u_j^{(k)} \\ -\Sigma\{b_{j,1}\beta_1^q(\alpha-\beta_1)+\cdots+b_{j,m-1}\beta_{m-1}^q(\alpha-\beta_{m-1})\}u_n^{(i)}\Big|\geq 1.$$

Here, let us put $\beta=\max\{|\beta_1|,\dots,|\beta_{m-1}|\}$, $A=\max\{|\alpha-\beta_i||i=1,2,\dots,m-1\}$, and $B=\max\{|b_{i,j}||i=1,2,\dots,m;\ j=0,1,\dots,m-1\}$. These values are independent of $u_m^{(k)}$, and $0<\beta<1$. Then we have $(m-1)AB\beta^p\{(2m-1)M_k+L_k\}\geq 1$, from which we can evaluate p as

$$\begin{aligned} p &\leq -[\log\{(2m-1)M_k+L_k\}+\log\{(m-1)AB\}]/\log\beta \\ &\leq C_1\log(L_k+C_2)+C_3 \end{aligned} \tag{12}$$

for some constants C_1, C_2, and C_3 independent of L_k and $u_m^{(k)}$.

On the other hand, from expression (6), we have

$$\begin{aligned} \left|t_q^{(1)}u_1^{(i)}+\cdots+t_q^{(m)}u_m^{(i)}\right| &\geq (b_{1,0}u_1^{(i)}+\cdots+b_{m,0}u_m^{(i)})\,\alpha^q-\Sigma\big(|b_{j,1}|+\cdots+|b_{j,m-1}|\big)u_j^{(i)} \\ &\geq (b_{1,0}u_1^{(1)}+\cdots+b_{m,0}u_m^{(1)})\,\alpha^q-m(m-1)M_k. \end{aligned}$$

We can find here a constant T depending only on the coefficients of (2) such that $t_n^{(i)}\leq T\alpha^n$ for $1\leq i\leq m$. We can also find an integer ν for which $t_n^{(m)}\geq(b_{m,0}/2)\alpha^n$ if $n\geq\nu$. Putting $U=\min\{b_{m,0}/2,\ \alpha^{-\nu}\}$, we have $t_n^{(m)}\geq U\alpha^n$ for $n\geq m$.

Using these inequalities and $t_p^{(m)}\geq 1$, we have the following evaluation, from (10),

$$\begin{aligned} L_k\geq u_m^{(k)} &\geq \left(t_q^{(1)}u_1^{(i)}+\cdots+t_q^{(m)}u_m^{(i)}\right)\Big/t_p^{(m)}-\left(t_p^{(1)}+\cdots+t_p^{(m-1)}\right)M_k\Big/t_p^{(m)} \\ &\geq \left(b_{1,0}u_1^{(i)}+\cdots+b_{m,0}u_m^{(i)}\right)\alpha^{q-p}\Big/T-(m-1)(mB+U^{-1}T)\,M_k. \end{aligned}$$

Taking the logarithms, we have

$$\begin{aligned} q-p &\leq \left[\log\{L_k+(m-1)(mB+U^{-1}T)M_k\}+\log T-\log\Sigma b_{j,0}u_j^{(i)}\right]\Big/\log\alpha \\ &= C_4\log(L_k+C_5)+C_6, \end{aligned}$$

which gives an evaluation of q together with inequality (12) as

$$q\geq C_1\log(L_k+C_2)+C_4\log(L_k+C_5)+C_7. \tag{13}$$

The constants C_4, C_5, C_6, and C_7 are independent of the choice of $u_m^{(k)}$ as well as the value of L_k.

Since we have already determined the $m-1$ initial terms of the k^{th} sequence, two different values of $u_m^{(k)}$ give different p^{th} terms that cannot coincide with the same $u_q^{(i)}$. Hence, the number N of the values of $u_m^{(k)}$ for which the resulting sequence $\{u_n^{(k)}\}$ has a common term with some of the first $k-1$ sequences will not exceed the number of triples (p,q,i) satisfying $u_p^{(k)}=u_q^{(i)}$. Evaluating the latter number using (12) and (13), we can obtain the inequality

$$N\leq(k-1)\{C_1\log(L_k+C_2)+C_3\}\{C_1\log(L_k+C_2)+C_4\log(L_k+C_5)+C_7\}\ll L_k-M_k$$

for large L_k.

Therefore, there must be a desired sequence $\{u_n^{(k)}\}$ in S disjoint with the $k-1$ sequences already built, and the proof can be completed by induction, since every natural number is contained in a sequence by the choice of $u_1^{(k)}$ determined in (8).

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AMS Classification Numbers: 11B37, 11B39

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