A NOTE ON A GENERAL CLASS OF POLYNOMIALS, PART II

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1. INTRODUCTION

In an earlier article [1] the author has discussed the properties of a set of polynomials $\{U_n(p,q;x)\}$ defined by

$$U_n(p,q;x) = (x+p)U_{n-1}(p,q;x) - qU_{n-2}(p,q;x),\ n \ge 2, \tag{1.1}$$

with $U_0(p,q;x) = 0$ and $U_1(p,q;x) = 1$.

Here and in the sequel the parameters p and q are arbitrary real numbers, and we denote by α, β the numbers such that $\alpha + \beta = p$ and $\alpha\beta = q$.

The aim of the present paper is to investigate the companion sequence of polynomials $\{V_n(p,q;x)\}$ defined by

$$V_n(p,q;x) = (x+p)V_{n-1}(p,q;x) - qV_{n-2}(p,q;x),\ n \ge 2, \tag{1.2}$$

with $V_0(p,q;x) = 2$ and $V_1(p,q;x) = x+p$.

The first few terms of the sequence $\{V_n(p,q;x)\}$ are

$$\begin{aligned}
V_2(p,q;x) &= (p^2-2q) + 2px + x^2,\\
V_3(p,q;x) &= (p^3-3pq) + (3p^2-3q)x + 3px^2 + x^3,\\
V_4(p,q;x) &= (p^4-4p^2q+2q^2) + (4p^3-8pq)x + (6p^2-4q)x^2 + 4px^3 + x^4.
\end{aligned}$$

We see by induction that there exists a sequence $\{d_{n,k}(p,q)\}_{\substack{n\ge1\\k\ge0}}$ of numbers such that

$$V_n(p,q;x) = \sum_{k\ge0} d_{n,k}(p,q)x^k,\ \underline{n \ge 1}, \tag{1.3}$$

with $d_{n,k}(p,q) = 0$ if $k \ge n+1$ and $d_{n,k}(p,q) = 1$ if $k = n$. For the sake of convenience, we define the sequence $\{d_{0,k}(p,q)\}$ by

$$d_{0,0}(p,q) = 1 \quad \text{and} \quad d_{0,k}(p,q) = 0 \text{ if } k \ge 1. \tag{1.4}$$

Notice that $V_0(p,q;x) = 2 = 2d_{0,0}(p,q)$.

Special cases of $\{V_n(p,q;x)\}$ which interest us are the Lucas polynomials $L_n(x)$ [2], the Pell-Lucas polynomials $Q_n(x)$ [7], the second Fermat polynomial sequence $\theta_n(x)$ [8], and the Chebyschev polynomials of the first kind $T_n(x)$ given by

$$\begin{aligned}
V_n(0,-1;x) &= L_n(x),\\
V_n(0,-1;2x) &= Q_n(x),\\
V_n(0,2;x) &= \theta_n(x),\\
V_n(0,1;2x) &= 2T_n(x).
\end{aligned} \tag{1.5}$$

Another interesting case is the Morgan-Voyce recurrence ([1], [5], [9], [10]. and [11]) given by $p=2$ and $q=1$ (or $\alpha=\beta=1$). In the sequel, we shall denote by $C_n(x)=V_n(2,1;x)$ this new kind of Morgan-Voyce polynomials, defined by

$$C_0(x)=2,\ C_1(x)=x+2,\ \text{and}\ C_n(x)=(x+2)C_{n-1}(x)-C_{n-2}(x),\ n\geq 2. \tag{1.6}$$

Remark 1.1: One can notice that $C_n(x^2)=L_{2n}(x)$. Actually, it is well known and readily proven that the sequence $\{L_{2n}(x)\}$ satisfies the recurrence relation $L_{2n}(x)=(x^2+2)L_{2n-2}(x)-L_{2n-4}(x)$, where $L_0(x)=2$ and $L_2(x)=x^2+2$. The result follows by this and (1.6).

It is clear that the sequence $\{V_n(p,q;0)\}$ is the generalized Lucas sequence defined by

$$V_n(p,q;0)=pV_{n-1}(p,q;0)-qV_{n-2}(p,q;0),\ n\geq 2,$$

with $V_0(p,q;0)=2$ and $V_1(p,q;0)=p$. Therefore, $V_n(p,q;0)=\alpha^n+\beta^n$. By (1.3), notice that

$$d_{n,0}(p,q)=V_n(p,q;0)=\alpha^n+\beta^n,\ \text{for}\ n\geq 1. \tag{1.8}$$

More generally, our aim is to express the coefficient $d_{n,k}(p,q)$ as a polynomial in (α,β) and as a polynomial in (p,q).

2. PRELIMINARIES

In this section we shall gather the results about polynomials $\{U_n(p,p;x)\}$ (1.1) which will be needed in the sequel. The reader may wish to consult [1].

Define the sequence $\{c_{n,k}(p,q)\}_{\substack{n\geq 0\\ k\geq 0}}$ by

$$U_{n+1}(p,q;x)=\sum_{k\geq 0}c_{n,k}(p,q)x^k, \tag{2.1}$$

where $c_{n,k}(p,q)=0$, for $k>n$. It was shown in [1] that

For every $n\geq 2$ and $k\geq 1$,

$$c_{n,k}(p,q)=pc_{n-1,k}(p,q)-qc_{n-2,k}(p,q)+c_{n-1,k-1}(p,q). \tag{2.2}$$

For every $n\geq 0$ and $k\geq 0$,

$$c_{n,k}(p,q)=\sum_{i+j=n-k}\binom{k+i}{k}\binom{k+j}{k}\alpha^i\beta^j. \tag{2.3}$$

If $p^2=4q$, then $\alpha=\beta=p/2$ and (2.3) becomes

$$c_{n,k}(p,q)=\binom{n+k+1}{2k+1}(p/2)^{n-k}. \tag{2.4}$$

If $p=0$, then $\alpha=-\beta=p$, $\alpha^2=-q$, and (2.3) becomes

$$\begin{cases} c_{n,n-2k}(0,q)=(-1)^k\binom{n-k}{k}q^k, & n-2k\geq 0,\\ c_{n,n-2k-1}(0,q)=0, & n-2k-1\geq 0. \end{cases} \tag{2.5}$$

For every $n \geq 0$ and $k \geq 0$,

$$c_{n,k}(p,q) = \sum_{r=0}^{[(n-k)/2]} (-1)^r \binom{n-r}{r}\binom{n-2r}{k} q^r p^{n-2r-k}. \tag{2.6}$$

The generating function of the sequence $\{U_n(p,q;x)\}$ is given by

$$f(p,q;x,t) = \sum_{n\geq 0} U_{n+1}(p,q;x)t^n = \frac{1}{1-(x+p)t+qt^2}. \tag{2.7}$$

The generating function $F_k(p,q;t)$ of the k^{th} column of coefficients $c_{n,k}(p,q)$ is given by

$$F_k(p,q;t) = \sum_{n\geq 0} c_{n+k,k}t^n = \frac{1}{(1-pt+qt^2)^{k+1}}. \tag{2.8}$$

For every $n \geq 0$, we have

$$U_{n+1}(p,q;0) = \sum_{r=0}^{[n/2]} (-1)^r \binom{n-r}{r} q^r p^{n-2r}. \tag{2.9}$$

3. THE TRIANGLE OF COEFFICIENTS

One can display the sequence $\{d_{n,k}(p,q)\}_{\substack{n\geq 0\\k\geq 0}}$ (1.3) in a triangle, thus,

TABLE 3.1

$n \backslash k$	0	1	2	3	4
0	1	0	0	0	0
1	p	1	0	0	0
2	p^2-2q	$2p$	1	0	0
3	p^3-3pq	$3p^2-3q$	$3p$	1	0
4	$p^4-4p^2q+2q^2$	$4p^3-8pq$	$6p^2-4q$	$4p$	1

For instance, the triangle of coefficients of the sequence $\{C_n(x)\}$ (1.6) is

TABLE 3.2

$n \backslash k$	0	1	2	3	4	5	6
0	1	0	0	0	0	0	0
1	2	1	0	0	0	0	0
2	2	4	1	0	0	0	0
3	2	9	6	1	0	0	0
4	2	16	20	8	1	0	0
5	2	25	50	35	10	1	0
6	2	36	105	112	54	12	1

Theorem 3.1: For every $n \geq 0$ and $k \geq 0$ we have

$$d_{n,k+1}(p,q) = \frac{1}{k+1}\frac{\partial d_{n,k}}{\partial p}.$$

Proof: One can suppose that $n \geq 1$ and it is clear by (1.2) that $V_n(p,q;x) = V_n(0,q;x+p)$. From this, we see that $V_n^{(k)}(p,q;x) = V_n^{(k)}(0,q;x+p)$, where the superscript in parentheses denotes the k^{th} derivative with respect to x. Thus, by Taylor's formula and (1.3),

$$d_{n,k}(p,q) = \frac{V_n^{(k)}(p,q;0)}{k!} = \frac{V_n^{(k)}(0,q;p)}{k!}. \tag{3.1}$$

Notice that these equalities are valid for every value of p. Now let us differentiate the first and the last member of (3.1) with respect to p (q being fixed) to get

$$\frac{\partial d_{n,k}}{\partial p} = \frac{V_n^{(k+1)}(0,q;p)}{k!} = (k+1)d_{n,k+1}(p,q).$$

The result can be checked against Table 3.1.

Remark 3.1: One can get the same result for the coefficient $c_{n,k}(p,q)$ (2.1), namely,

$$\frac{\partial c_{n,k}}{\partial p} = (k+1)c_{n,k+1}(p,q).$$

Comparing the coefficients of x^k in the two members of (1.3), we see by (1.2) that, for $n \geq 2$ and $k \geq 1$,

$$d_{n,k}(p,q) = d_{n-1,k-1}(p,q) + pd_{n-1,k}(p,q) - qd_{n-2,k}(p,q), \tag{3.2}$$

which is a relation similar to (2.2). From this, one can obtain another recurrence relation.

Theorem 3.2: For every $n \geq 1$ and $k \geq 1$, we have

$$\begin{aligned} d_{n,k}(p,q) &= \beta d_{n-1,k}(p,q) + \sum_{i=0}^{n-1} \alpha^{n-i-1} d_{i,k-1}(p,q) \\ &= \alpha d_{n-1,k}(p,q) + \sum_{i=0}^{n-1} \beta^{n-1-i} d_{i,k-1}(p,q). \end{aligned} \tag{3.3}$$

Proof: In fact, (3.3) is clear by direct computation for $n \leq 2$ [recall that $d_{0,0}(p,q) = 1$ and that $\alpha + \beta = p$]. Using (3.2), we see that the end of the proof is analogous to the proof of Theorem 1 in [1].

For instance, in the case of the Morgan-Voyce polynomial $C_n(x)$ (1.6) we have $\alpha = \beta = 1$, and (3.2) becomes (see Table 3.2)

$$d_{n,k}(2,1) = d_{n-1,k}(2,1) + \sum_{i=0}^{n-1} d_{i,k+1}(2,1),$$

which is the recursive definition of the DFF and DFFz triangles (see [3], [4], [5]) known to be the triangle of coefficients of the usual Morgan-Voyce polynomials.

4. DETERMINATION OF $d_{n,k}(p,q)$ AS A POLYNOMIAL IN (α, β)

The determination of $d_{n,k}(p,q)$ will proceed easily from the following lemmas. The first of these is a well-known result on second-order recurring sequences that can be proven by induction using (1.1) and (1.2).

Lemma 4.1: For every $n \geq 1$, we have

$$V_n(p,q;x) = U_{n+1}(p,q;x) - qU_{n-1}(p,q;x). \tag{4.1}$$

Lemma 4.2: For every $n \geq 0$, we have

$$V_n'(p,q;x) = nU_n(p,q;x), \tag{4.2}$$

where the prime represents the first derivative w.r.t. x.

Proof: By (1.1) and (1.2), the result is clear if $n = 0$ or $n = 1$. Assuming the result is true for $n \geq 1$, we obtain by (1.2),

$$\begin{aligned} V_{n+1}'(p,q;x) &= (x+p)V_n'(p,q;x) - qV_{n-1}'(p,q;x) + V_n(p,q;x) \\ &= n[(x+p)U_n(p,q;x) - qU_{n-1}(p,q;x)] + V_n(p,q;x) + qU_{n-1}(p,q;x) \\ &= nU_{n+1}(p,q;x) + U_{n+1}(p,q;x) \quad \text{by (1.1) and (4.1),} \\ &= (n+1)U_{n+1}(p,q;x). \end{aligned}$$

This concludes the proof of Lemma 4.2.

Lemma 4.3: For every $n \geq 1$ and $k \geq 1$, we have

$$d_{n,k}(p,q) = \frac{n}{k} c_{n-1,k-1}(p,q). \tag{4.3}$$

Proof: Comparing the coefficients of x^{k-1} in the two members of (4.2) we see by (1.3) and (2.1) that

$$kd_{n,k}(p,q) = nc_{n-1,k-1}(p,q), \; n \geq 1, k \geq 1.$$

Lemma 4.3 and (2.3) yield

Theorem 4.1: For every $n \geq 1$ and $k \geq 1$, we have

$$d_{n,k}(p,q) = \frac{n}{k} \sum_{i+j=n-k} \binom{k+i-1}{k-1} \binom{k+j-1}{k-1} \alpha^i \beta^j. \tag{4.4}$$

Remark 4.1: Recall from (1.8) that $d_{n,0}(p,q) = \alpha^n + \beta^n$ (for $n > 0$), an expression which can be compared with (4.4).

Let us examine two particular cases.

(i) Firstly, supposing that $p^2 = 4q$ (or $\alpha = \beta = p/2$), then by (2.4) we see that equation (4.3) becomes

$$\begin{aligned} d_{n,k}(p,q) &= \frac{n}{k}\binom{n+k-1}{2k-1}(p/2)^{n-k},\ n \geq 1, k \geq 1, \\ &= \frac{2n}{n+k}\binom{n+k}{2k}(p/2)^{n-k}. \end{aligned} \tag{4.5}$$

Notice that this last expression is again valid if $k = 0$, since $d_{n,0}(p,q) = \alpha^n + \beta^n = 2(p/2)^n$. We also see that $d_{n,1}(p,q) = n^2(p/2)^{n-1}$ (see Table 3.2, where $p = 2$). For instance, the decomposition of the polynomial $C_n(x)$ (1.6) is given by

$$\begin{aligned} C_n(x) &= 2 + \sum_{k=1}^{n} \frac{n}{k}\binom{n+k-1}{2k-1}x^k, \text{ for } n \geq 1, \\ &= 2\sum_{k=0}^{n} \frac{n}{n+k}\binom{n+k}{2k}x^k. \end{aligned}$$

(ii) Secondly, supposing that $p = 0$, we have $\alpha = -\beta$, $q = -\alpha^2$, and by (2.5) we see that equation (4.3) becomes, for $n \geq 1$,

$$\begin{aligned} d_{n,n-2k}(0,q) &= \frac{n}{n-2k}(-1)^k\binom{n-1-k}{k}q^k \\ &= \frac{n}{n-k}(-1)^k\binom{n-k}{k}q^k, \text{ for } n-2k \geq 1. \end{aligned} \tag{4.6}$$

Notice that the last member is again defined for $n-2k = 0$ $(k \geq 1)$ with value $2(-1)^k q^k$. Now, by Remark 4.1, we get that

$$d_{2k,0}(0,q) = \alpha^{2k} + \beta^{2k} = 2(-1)^k q^k, \text{ for } k \geq 1.$$

We deduce from these remarks that (4.6) is again true if $n = 2k$ $(k \geq 1)$. On the other hand, we see by (2.5) that equation (4.3) becomes

$$d_{n,n-2k-1}(0,q) = 0, \text{ for } n-2k-1 \geq 1. \tag{4.7}$$

Now by Remark 4.1 we have

$$d_{2k+1,0}(0,q) = \alpha^{2k+1} + \beta^{2k+1} = 0, \text{ for } k \geq 0.$$

We deduce from these remarks that (4.7) is again true if $n-2k-1 = 0$ $(k \geq 0)$. Now, by (1.3),

$$\begin{aligned} V_n(0,q;x) &= \sum_{k=0}^{n} d_{n,k}(0,q)x^k = \sum_{k=0}^{n} d_{n,n-k}(0,q)x^{n-k} \\ &= \sum_{k=0}^{[n/2]} d_{n,n-2k}(0,q)x^{n-2k}. \end{aligned}$$

Thus, by (4.6) and (4.7) we get

$$V_n(0,q;x) = \sum_{k=0}^{[n/2]} (-1)^k \frac{n}{n-k}\binom{n-k}{k}q^k x^{n-2k}, \text{ for } n \geq 1. \tag{4.8}$$

If $p = 0$ and $q = -1$, we obtain the known decomposition of Lucas polynomials $L_n(x)$ and of Pell-Lucas polynomials $Q_n(x) = L_n(2x)$ (see [7]), namely,

$$L_n(x) = \sum_{k=0}^{[n/2]} \frac{n}{n-k}\binom{n-k}{k} x^{n-2k}, \text{ for } n \geq 1.$$

The reader can also obtain similar formulas for the Chebyschev polynomials of the first kind ($p = 0, q = 1$), and for the second Fermat polynomial sequence ($p = 0, q = 2$).

5. DETERMINATION OF $d_{n,k}(p, q)$ AS A POLYNOMIAL IN (p, q)

Theorem 5.1: For every $n \geq 1$ and $k \geq 0$, we have

$$d_{n,k}(p, q) = \sum_{r=0}^{[(n-k)/2]} (-1)^r \frac{n}{n-r}\binom{n-r}{r}\binom{n-2r}{k} q^r p^{n-2r-k}. \tag{5.1}$$

Proof: By (3.1) we know that

$$d_{n,k}(p, q) = \frac{V_n^{(k)}(0, q; p)}{k!},$$

and by (4.8) one can express the right member as

$$\sum_{r=0}^{[n/2]} (-1)^r \frac{n}{n-r}\binom{n-r}{r} q^r \frac{(n-2r)\cdots(n-2r-k+1)}{k!} p^{n-2r-k}$$
$$= \sum_{r=0}^{[(n-k)/2]} (-1)^r \frac{n}{n-r}\binom{n-r}{r}\binom{n-2r}{k} q^r p^{n-2r-k}.$$

This completes the proof of Theorem 5.1.

Remark 5.1: If $k = 0$, we get by (1.8) the known Waring formula, namely,

$$\alpha^n + \beta^n = \sum_{r=0}^{[n/2]} (-1)^r \frac{n}{n-r}\binom{n-r}{r} (\alpha\beta)^r (\alpha+\beta)^{n-2r}, \text{ for } n \geq 1.$$

6. GENERATING FUNCTIONS

Define the generating function of the sequence $\{V_n(p, q; x)\}$ by

$$g(p, q; x, t) = V_0(p, q; x)/2 + \sum_{n\geq 1} V_n(p, q; x) t^n. \tag{6.1}$$

For brevity, we put $g(p, q; x, t) = g(x, t)$ and $V_n(p, q; x) = V_n(x)$. By (6.1) and (1.2) we get, since $V_0(x) = 2$ and $V_1(x) = x + p$,

$$g(x, t) = 1 + (x+p)t + (x+p)t\sum_{n\geq 2} V_{n-1}(x)t^{n-1} - qt^2\sum_{n\geq 2} V_{n-2}(x)t^{n-2}$$
$$= 1 + (x+p)t + (x+p)t[g(x, t) - 1] - qt^2[g(x, t) + 1],$$

and from this we deduce easily that

$$g(x, t) = \frac{1 - qt^2}{1 - (x+p)t + qt^2}. \tag{6.2}$$

Let us define now the generating function of the k^{th} column of the triangle $d_{n,k}(p,q)$ in Table 3.1 by

$$G_k(p,q;t) = \sum_{n\geq 0} d_{n+k,k}(p,q)t^n, \ k \geq 0. \tag{6.3}$$

From (6.2), one can obtain a closed expression for the function G_k, namely,

Theorem 6.1: For every $k \geq 0$, we have

$$G_k(p,q;t) = \frac{1-qt^2}{(1-pt+qt^2)^{k+1}}. \tag{6.4}$$

Proof: For brevity, we omit parameters p and q in expressions for $g(p,q;x,t)$, $V_n(p,q;x)$, $d_{n,k}(p,q)$, and $G_k(p,q;t)$. If $k=0$, we have by (6.3), (1.3), and (1.4)

$$\begin{aligned} G_0(t) &= \sum_{n\geq 0} d_{n,0}t^n = 1 + \sum_{n\geq 1} V_n(0)t^n \\ &= g(0,t) = \frac{1-qt^2}{1-pt+qt^2}, \text{ by (6.2).} \end{aligned}$$

Assuming now that $k \geq 1$, (6.1) and (6.2) yield

$$\frac{k!\,t^k(1-qt^2)}{(1-(x+p)t+qt^2)^{k+1}} = \frac{\partial^k}{\partial x^k} g(x,t) = \sum_{n\geq 1} V_n^{(k)}(x)t^n = \sum_{n\geq 0} V_{n+k}^{(k)}(x)t^{n+k},$$

since $V_n(x)$ is a polynomial of degree n.

Put $x=0$ in the last formula and recall that $d_{n+k,k} = \frac{V_{n+k}^{(k)}(0)}{k!}$ by (1.3) and Taylor's formula, to obtain

$$\frac{1-qt^2}{(1-pt+qt^2)^{k+1}} = \sum_{n\geq 0} d_{n+k,k}t^n = G_k(t).$$

Hence, the theorem.

Formulas (6.2) and (6.4) can be compared with (2.7) and (2.8).

7. RISING DIAGONAL FUNCTIONS

Define the rising diagonal functions $\Pi_n(p,q;x)$ of the sequence $\{d_{n,k}(p,q)\}$ by

$$\Pi_n(p,q;x) = \sum_{k=0}^{n} d_{n-k,k}(p,q)x^k = \sum_{k=0}^{[n/2]} d_{n-k,k}(p,q)x^k, \ n \geq 1. \tag{7.1}$$

From Table 3.1, notice that

$$\Pi_1(x) = p, \ \Pi_2(x) = (p^2-2q)+x, \text{ and } \Pi_3(x) = (p^3-3pq)+2px, \tag{7.2}$$

where, for brevity, we put $\Pi_n(x)$ for $\Pi_n(p,q;x)$.

Theorem 7.1: For every $n \geq 3$, we have

$$\Pi_n(x) = p\Pi_{n-1}(x) + (x-q)\Pi_{n-2}(x). \tag{7.3}$$

Proof: By (7.2), the statement holds for $n = 3$. Supposing the result is true for $n \geq 3$, we get by (7.1),

$$\Pi_{n+1}(x) = d_{n+1,0} + \sum_{k=1}^{[(n+1)/2]} d_{n+1-k,k}x^k.$$

Recall from (1.2) and (1.8) that $d_{n+1,0} = V_{n+1}(0) = pd_{n,0} - qd_{n-1,0}$ and notice that $n+1-k \geq n+1-[(n+1)/2] \geq 2$, since $n \geq 3$. By these remarks and (3.2), one can see that

$$\begin{aligned}\Pi_{n+1}(x) &= pd_{n,0} - qd_{n-1,0} + \sum_{k=1}^{[(n+1)/2]} (d_{n-k,k-1} + pd_{n-k,k} - qd_{n-1-k,k})x^k \\ &= p\sum_{k=0}^{[(n+1)/2]} d_{n-k,k}x^k - q\sum_{k=0}^{[(n+1)/2]} d_{n-1-k,k}x^k + x\sum_{k=0}^{[(n+1)/2]-1} d_{n-1-k,k}x^k \\ &= p\Pi_n(x) + (x-q)\Pi_{n-1}(x),\end{aligned}$$

since $[(n+1)/2]-1 = [(n-1)/2]$. Hence, the theorem.

Corollary 7.1: For every $n \geq 1$, we have

$$\Pi_n(p,q;x) = U_{n+1}(p,q-x;0) - qU_{n-1}(p,q-x;0). \tag{7.4}$$

Proof: By (1.1) the sequence $\{U_n(p,q-x;0)\}$ satisfies the recurrence (7.3) with

$$U_0(p,q-x;0) = 0,\ U_1(p,q-x;0) = 1,\ U_2(p,q-x;0) = p,\ U_3(p,q-x;0) = (p^2-q)+x.$$

From this and (7.2), it is readily verified that (7.4) holds for $n = 1$ and $n = 2$, and the conclusion follows since the two members of (7.4) satisfy recurrence (7.3).

Corollary 7.2: For every $n \geq 1$, we have

$$\Pi_n(x) = \binom{n-[n/2]}{[n/2]}p^{n-2[n/2]}(x-q)^{[n/2]} + \sum_{r=0}^{[(n-2)/2]} p^{n-2-2r}(x-q)^r\left[\binom{n-r}{r}p^2 - \binom{n-2-r}{r}q\right].$$

Proof: From (2.9), we get that

$$U_{n+1}(p,q-x;0) = \sum_{r=0}^{[n/2]}\binom{n-r}{r}(x-q)^r p^{n-2r},$$

and the result follows by this and Corollary 7.1.

Let us examine two particular cases.

(i) If $x = q$, then by (7.1)

$$\Pi_n(p,q;q) = \sum_{k=0}^{[n/2]} d_{n-k,k}(p,q)q^k = p^{n-2}(p^2-q),\ \text{for } n \geq 2.$$

For instance, if $p = 2$ and $q = 1$ [Morgan-Voyce polynomial $C_n(x)$ (1.6)], we get

$$\sum_{k=0}^{[n/2]} d_{n-k,k}(2,1) = 3\cdot 2^{n-2},\ n \geq 2.$$

(ii) If $p = 0$, then

$$\Pi_{2m}(0,q;x) = \sum_{k=0}^{m} d_{2m-k,k}(0,q)x^k = (x-q)^{m-1}(x-2q), \text{ for } m \geq 1.$$

For instance, if $p = 0$ and $q = 1$ (Chebyschev polynomials of the first kind), or if $p = 0$ and $q = 2$ (second Fermat polynomials), this identity, with slightly different notations, was noticed by Horadam [8].

8. ORTHOGONALITY OF THE SEQUENCE $\{V_n(p,q;x)\}$

In this section we shall suppose that $q > 0$. Consider the sequence $\{W_n(p,q;x)\}$ defined by

$$W_n(p,q;x) = 2q^{n/2}T_n\left(\frac{x+p}{2\sqrt{q}}\right), \tag{8.1}$$

where $T_n(x)$ is the n^{th} Chebyschev polynomial of the first kind. Notice that

$$\begin{cases} W_0(p,q;x) = 2, \\ W_1(p,q;x) = x+p. \end{cases} \tag{8.2}$$

The recurrence relation of Chebyschev polynomials yields, for $n \geq 2$,

$$\begin{aligned} W_n(p,q;x) &= 2q^{n/2}\left[\left(\frac{x+p}{\sqrt{q}}\right)T_{n-1}\left(\frac{x+p}{2\sqrt{q}}\right) - T_{n-2}\left(\frac{x+p}{2\sqrt{q}}\right)\right] \\ &= (x+p)\left[2q^{(n-1)/2}T_{n-1}\left(\frac{x+p}{2\sqrt{q}}\right)\right] - q\left[2q^{(n-2)/2}T_{n-2}\left(\frac{x+p}{2\sqrt{q}}\right)\right] \\ &= (x+p)W_{n-1}(p,q;x) - qW_{n-2}(p,q;x). \end{aligned} \tag{8.3}$$

From (8.2) and (8.3), we get that

$$W_n(p,q;x) = V_n(p,q;x), \text{ for } n \geq 0. \tag{8.4}$$

Recalling that the sequence $\{T_n(x)\}$ is orthogonal over $[-1,+1]$ with respect to the weight $(1-x^2)^{-1/2}$, we deduce from this that the sequence $\{V_n(p,q;x)\}$ is orthogonal over $[-p-2\sqrt{q}, -p+2\sqrt{q}]$ with respect to the weight $w(x) = (-x^2-2px-\Delta)^{-1/2}$, where $\Delta = p^2-4q$. The proof is similar to that in [1], Section 7.

If $\omega = \cos t$ $(0 \leq t \leq \pi)$, it is well known that $T_n(\omega) = \cos nt$. Thus, by (8.1) and (8.4) we have

$$V_n(p,q;-p+2\omega\sqrt{q}) = 2q^{n/2}T_n(\omega) = 2q^{n/2}\cos nt.$$

Hence, we see that the roots of $V_n(p,q;x)$ are given by

$$x_k = -p+2\sqrt{q}\cos\left(\frac{(2k+1)\pi}{2n}\right), \; n \geq 1; \; k = 0, \ldots, (n-1).$$

For instance, the roots of the Morgan-Voyce polynomial $C_n(x)$ (1.6) are

$$x_k = -2+2\cos\left(\frac{(2k+1)\pi}{2n}\right) = -4\sin^2\left(\frac{(2k+1)\pi}{4n}\right), \; k = 0, \ldots, (n-1).$$

By Remark 1.1 we know that $C_n(x^2) = L_{2n}(x)$. Thus, the roots of $L_{2n}(x)$ are given by (see [6])

$$x'_k = \pm 2i \sin\left(\frac{(2k+1)\pi}{4n}\right),\ k = 0, \ldots, (n-1),$$

where $i = \sqrt{-1}$. On the other hand, the roots of the second Fermat polynomial $\theta_n(x) = V_n(0,2;\, x)$ are

$$x_k = 2\sqrt{2}\cos\left(\frac{(2k+1)\pi}{2n}\right),\ k = 0, \ldots, (n-1).$$

9. CONCLUDING REMARK

In a future paper we shall investigate the differential properties of the sequences $\{U_n(p,q;\, x)\}$ and $\{V_n(p,q;\, x)\}$.

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