DIFFERENTIAL PROPERTIES OF A GENERAL CLASS OF POLYNOMIALS

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(*Submitted April 1994*)

1. INTRODUCTION

Let us consider the generalized Fibonacci polynomials $U_n(p,q;x)$ and the generalized Lucas polynomials $V_n(p,q;x)$ (or simply U_n and V_n if there is no danger of confusion) defined by

$$U_n = (x+p)U_{n-1} - qU_{n-2} \quad (U_0 = 0, U_1 = 1), \tag{1.1}$$

and

$$V_n = (x+p)V_{n-1} - qV_{n-2} \quad (V_0 = 2, V_1 = x+p). \tag{1.2}$$

The parameters p and q as well as the variable x are arbitrary real numbers and we denote by $\alpha = \alpha(x)$ and $\beta = \beta(x)$ the numbers such that $\alpha + \beta = x + p$ and $\alpha\beta = q$. The polynomials U_n and V_n can be expressed by means of the Binet forms

$$U_n = \frac{\alpha^n - \beta^n}{\Delta^{1/2}}, \quad \text{for } \Delta \neq 0, \tag{1.3}$$

and

$$V_n = \alpha^n + \beta^n, \tag{1.4}$$

where

$$\Delta = \Delta(x) = (x+p)^2 - 4q. \tag{1.5}$$

Recall that

$$\alpha = ((x+p) + \Delta^{1/2})/2, \quad \beta = ((x+p) - \Delta^{1/2})/2. \tag{1.6}$$

Notice that $\Delta > 0$ for every x if $q < 0$ for all x sufficiently large if $q \geq 0$.

Particular cases of $U_n(p,q;x)$ and $V_n(p,q;x)$ are the Fibonacci and Lucas polynomials ($F_n(x)$ and $L_n(x)$), the Pell and Pell-Lucas polynomials [6] ($P_n(x)$ and $Q_n(x)$), the first and the second Fermat polynomials [7] ($\Phi_n(x)$ and $\Theta_n(x)$), the Morgan–Voyce polynomials [1, 2, 5, 8, 9, 10] ($B_n(x)$ and $C_n(x)$), and the Chebyschev polynomials ($S_n(x)$ and $T_n(x)$) given by

$$\begin{array}{ll} U_n(0,-1;x) = F_n(x), & V_n(0,-1;x) = L_n(x) \\ U_n(0,-1;2x) = P_n(x), & V_n(0,-1;2x) = Q_n(x) \\ U_n(0,2;x) = \Phi_n(x), & V_n(0,2;x) = \Theta_n(x) \\ U_{n+1}(2,1;x) = B_n(x), & V_n(2,1;x) = C_n(x) \\ U_n(0,1;2x) = S_n(x), & V_n(0,1;2x) = 2T_n(x). \end{array} \tag{1.7}$$

In earlier papers [1, 2] the author has discussed the combinatorial properties of the coefficients of U_n and V_n. Here, we shall investigate the differential properties satisfied by these polynomials, such as differential equations and Rodrigues' formulas.

Let us define the sequence $\{c_{n,k}\}_{n\geq k\geq 0}$ by

$$c_{n,0} = 2\frac{n!}{(2n)!}, \quad n \geq 0, \tag{1.8}$$

and

$$c_{n,k} = 2\frac{n!}{(2n)!}\frac{n}{n+k}\frac{(n+k)!}{(n-k)!}, \quad n \geq k \geq 1. \tag{1.9}$$

Notice that

$$c_{n,k+1} = (n^2 - k^2)c_{n,k}, \quad n \geq k+1 \geq 1. \tag{1.10}$$

Our main results are the following theorems.

Theorem 1: For every real number x, the polynomial

$$U_n^{(k-1)} = \frac{d^{k-1}}{dx^{k-1}}U_n, \quad k \geq 1,$$

and the polynomial

$$V_n^{(k)} = \frac{d^k}{dx^k}V_n, \quad k \geq 0,$$

satisfy the differential equation $E_{n,k}$:

$$\Delta z'' + (2k+1)(x+p)z' + (k^2 - n^2)z = 0. \tag{1.11}$$

Theorem 2: For every x such that $\Delta > 0$, we have

$$U_n = nc_{n,0}\Delta^{-1/2}\frac{d^{n-1}}{dx^{n-1}}\Delta^{n-1/2}, \quad n \geq 1, \tag{1.12}$$

and

$$V_n = c_{n,0}\Delta^{1/2}\frac{d^n}{dx^n}\Delta^{n-1/2}, \quad n \geq 0, \tag{1.13}$$

where $c_{n,0}$ is defined by (1.8).

More generally, we also have Rodrigues' formulas for $U_n^{(k)}$ and $V_n^{(k)}$, namely,

Theorem 3: For every x such that $\Delta > 0$ and every $k \geq 0$, we have

$$U_n^{(k)} = \frac{(n^2 - k^2)}{n}c_{n,k}\Delta^{-k-1/2}\frac{d^{n-k-1}}{dx^{n-k-1}}\Delta^{n-1/2}, \quad n \geq k+1, \tag{1.14}$$

and

$$V_n^{(k)} = c_{n,k}\Delta^{-k+1/2}\frac{d^{n-k}}{dx^{n-k}}\Delta^{n-1/2}, \quad n \geq k, \tag{1.15}$$

where $c_{n,k}$ is defined by (1.9).

Notice that Theorem 3 reduces to Theorem 2 for $k = 0$ and that (1.14) can be written, by (1.10),

$$U_n^{(k)} = \frac{c_{n,k+1}}{n}\Delta^{-k-1/2}\frac{d^{n-k-1}}{dx^{n-k-1}}\Delta^{n-1/2}, \quad n \geq k. \tag{1.16}$$

2. PROOF OF THEOREM 1

It is readily proven [3, 4] by (1.5) and (1.6) that, for every x such that $\Delta > 0$,

$$\begin{cases} \alpha' = \alpha\Delta^{-1/2}, \\ \beta' = -\beta\Delta^{-1/2}, \end{cases} \tag{2.1}$$

and thus that

$$\begin{cases} (\alpha^n)' = n\alpha^n\Delta^{-1/2}, \\ (\beta^n)' = -n\beta^n\Delta^{-1/2}. \end{cases} \tag{2.2}$$

By this, (1.3), and (1.4), we see [3, 4] that

$$V_n' = nU_n \tag{2.3}$$

and therefore that

$$V_n^{(k)} = nU_n^{(k-1)}, \quad k \geq 1. \tag{2.4}$$

Notice that these identities are valid for every value of x, and not only when $\Delta > 0$, since the two members are polynomials. By (2.2), we also deduce that α^n and β^n, whence $V_n = \alpha^n + \beta^n$ satisfies the differential equation

$$\frac{d}{dx}(\Delta^{1/2}y') = n^2\Delta^{-1/2}y, \quad \text{for } \Delta > 0, \tag{2.5}$$

which is equivalent, for $\Delta > 0$, to the equation $E_{n,0}$ [see (1.11)], namely,

$$\Delta y'' + (x+p)y' - n^2y = 0. \tag{2.6}$$

Notice that V_n satisfies $E_{n,0}$ for every value of x, since, in that case, the first member of (2.6) is a polynomial.

Differentiating (2.6) k times and using Leibniz' rule, we see that $z = y^{(k)}$ satisfies the differential equation $E_{n,k}$ (1.11). Hence, $E_{n,k}$ is satisfied by $V_n^{(k)}, k \geq 0$, and $U_n^{(k-1)} = \frac{1}{n}V_n^{(k)}, \ k \geq 1$. This concludes the proof.

For instance, the Morgan–Voyce polynomial $B_n(x) = U_{n+1}(2,1;x)$ satisfies the differential equation $E_{n+1,1}$

$$x(x+4)z'' + 3(x+2)z' - n(n+2)z = 0.$$

This result was first noticed by Swamy [10].

Remark: When $\Delta > 0$, it is easy to verify that $E_{n,k}$ can be written as

$$\frac{d}{dx}[\Delta^{k+1/2}z'] = (n^2 - k^2)\Delta^{k-1/2}z, \tag{2.7}$$

which is a generalization of (2.5).

We now give another (nonpolynomial) solution of $E_{n,k}$.

Proposition 1: Let n and k be two integers such that $n+k-1 \geq 0$. Then, for $\Delta > 0$, the function $\frac{d^{n+k-1}}{dx^{n+k-1}}\Delta^{n-1/2}$ is a solution of $E_{n,k}$.

Proof: It is easy to verify that, for $\Delta > 0$, $\Delta^{n-1/2}$ is a solution of the differential equation

$$\Delta y'' - (2n-3)(x+p)y' - (2n-1)y = 0. \tag{2.8}$$

Differentiating (2.8) $(n+k-1)$ times and putting $z = y^{(n+k-1)}$, we obtain

$$\Delta z'' + 2\binom{n+k-1}{1}(x+p)z' + 2\binom{n+k-1}{2}z - (2n-3)\left[(x+p)z' + \binom{n+k-1}{1}z\right] - (2n-1)z = 0.$$

After some rearrangement, one can see that this equation is identical to $E_{n,k}$.

Remark: Using the formulation (2.7) of $E_{n,k}$ and putting $z = \frac{d^{n+k-1}}{dx^{n+k-1}}\Delta^{n-1/2}$, one can write

$$\frac{d}{dx}\left[\Delta^{k+1/2}\frac{d^{n+k}}{dx^{n+k}}\Delta^{n-1/2}\right] = (n^2-k^2)\Delta^{k-1/2}\frac{d^{n+k-1}}{dx^{n+k-1}}\Delta^{n-1/2}. \tag{2.9}$$

Changing k to $(-k-1)$ in (2.9), where $n-k \geq 2$, we obtain a formula that we shall need later:

$$\frac{d}{dx}\left[\Delta^{-k-1/2}\frac{d^{n-k-1}}{dx^{n-k-1}}\Delta^{n-1/2}\right] = (n^2-(k+1)^2)\Delta^{-k-3/2}\frac{d^{n-k-2}}{dx^{n-k-2}}\Delta^{n-1/2}. \tag{2.10}$$

In particular, changing n to $(n+1)$, and putting $k=-1$, we get

$$\frac{d}{dx}\left[\Delta^{1/2}\frac{d^{n+1}}{dx^{n+1}}\Delta^{n+1/2}\right] = (n+1)^2\Delta^{-1/2}\frac{d^{n}}{dx^{n}}\Delta^{n+1/2}, \quad n \geq 0. \tag{2.11}$$

3. PROOF OF THEOREM 2

In the proof of Theorem 2, we shall need the following well-known and readily proven result:

$$V_{n+1} = \frac{1}{2}[(x+p)V_n + \Delta U_n]. \tag{3.1}$$

By (1.8), formula (1.12) (resp. (1.13)) is clear if $n=1$ (resp. $n=0$ or $n=1$). Supposing that (1.12) and (1.13) are true for $n \geq 1$, we get by (3.1) that

$$V_{n+1} = \frac{n!}{(2n!)}\Delta^{1/2}\left[(x+p)\frac{d^n}{dx^n}\Delta^{n-1/2} + n\frac{d^{n-1}}{dx^{n-1}}\Delta^{n-1/2}\right]. \tag{3.2}$$

On the other hand, one can notice by (1.5) that

$$\begin{aligned}\frac{d^{n+1}}{dx^{n+1}}\Delta^{n+1/2} &= \frac{d^n}{dx^n}\left[(2n+1)(x+p)\Delta^{n-1/2}\right] \\ &= (2n+1)\left[(x+p)\frac{d^n}{dx^n}\Delta^{n-1/2} + n\frac{d^{n-1}}{dx^{n-1}}\Delta^{n-1/2}\right].\end{aligned} \tag{3.3}$$

From (3.2) and (3.3), we see that

$$V_{n+1} = \frac{n!}{(2n!)}\Delta^{1/2}\frac{1}{2n+1}\frac{d^{n+1}}{dx^{n+1}}\Delta^{n+1/2} = 2\frac{(n+1)!}{(2n+2)!}\Delta^{1/2}\frac{d^{n+1}}{dx^{n+1}}\Delta^{n+1/2}, \tag{3.4}$$

which is the needed formula for V_{n+1}.

Now we see, by (2.3) and (3.4), that

$$U_{n+1} = \frac{1}{n+1}V'_{n+1} = 2\frac{n!}{(2n+2)!}\frac{d}{dx}\left[\Delta^{1/2}\frac{d^{n+1}}{dx^{n+1}}\Delta^{n+1/2}\right]$$
$$= 2\frac{n!}{(2n+2)!}(n+1)^2\Delta^{-1/2}\frac{d^n}{dx^n}\Delta^{n+1/2}, \quad \text{by (2.11)}, \tag{3.5}$$
$$= 2(n+1)\frac{(n+1)!}{(2n+2)!}\Delta^{-1/2}\frac{d^n}{dx^n}\Delta^{n+1/2}.$$

This completes the proof of Theorem 2.

4. PROOF OF THEOREM 3

We proceed by induction on k. By Theorem 2, statement (1.14) clearly holds for $k = 0$ and every $n \geq 1$. Supposing that (1.14) holds for $k \geq 0$ and every $n \geq k+1$ we get, by (1.16),

$$U_n^{(k+1)} = \frac{d}{dx}U_n^{(k)} = \frac{c_{n,k+1}}{n}\frac{d}{dx}\left[\Delta^{-k-1/2}\frac{d^{n-k-1}}{dx^{n-k-1}}\Delta^{n-1/2}\right],$$

and, by (2.10), we have at once that

$$U_n^{(k+1)} = \frac{c_{n,k+1}}{n}\left[n^2-(k+1)^2\right]\Delta^{-k-3/2}\frac{d^{n-k-2}}{dx^{n-k-2}}\Delta^{n-1/2}, \quad n \geq k+2,$$

which is the needed formula for $U_n^{(k+1)}$.

On the other hand, statement (1.15) holds for $k = 0$, by Theorem 2. When $k \geq 1$ we get, by (2.4) and (1.14) that

$$V_n^{(k)} = nU_n^{(k-1)} = c_{n,k}\Delta^{-k+1/2}\frac{d^{n-k}}{dx^{n-k}}\Delta^{n-1/2}, \quad n \geq k.$$

This completes the proof of Theorem 3.

REFERENCES

1. R. André-Jeannin. "A Note on a General Class of Polynomials." *The Fibonacci Quarterly* **32.5** (1994):445-54.
2. R. André-Jeannin. "A Note on a General Class of Polynomials, Part II." *The Fibonacci Quarterly* **33.4** (1995):341-51.
3. P. Filipponi & A. F. Horadam. "Derivative Sequences of Fibonacci and Lucas Polynomials." In *Applications of Fibonacci Numbers* **4**:99-108. Ed. G. E. Bergum, A. N. Philippou, & A. F. Horadam. Dordrecht: Kluwer, 1991.
4. P. Filipponi & A. F. Horadam. "Second Derivative Sequences of Fibonacci and Lucas Polynomials." *The Fibonacci Quarterly* **31.3** (1993):194-204.
5. G. Ferri, M. Faccio, & A. D'Amico. "The DFF and DFFz Triangles and Their Mathematical Properties." In *Applications of Fibonacci Numbers* **5.** Ed. A. N. Philippou, G. E. Bergum, & A. F. Horadam. Dordrecht: Kluwer, 1994.
6. A. F. Horadam & Br. J. M. Mahon. "Pell and Pell-Lucas Polynomials." *The Fibonacci Quarterly* **23.1** (1985):7-20.

7. A. F. Horadam. "Chebyschev and Fermat Polynomials for Diagonal Functions." *The Fibonacci Quarterly* **19.4** (1979):328-33.
8. J. Lahr. "Fibonacci and Lucas Numbers and the Morgan–Voyce Polynomials in Ladder Networks and in Electrical Line Theory." In *Applications of Fibonacci Numbers* (AU? Please check Volume No.), pp. 141-61. Ed. A. N. Philippou, G. E. Bergum, & A. F. Horadam. Dordrecht: Kluwer, 1986.
9. M. N. S. Swamy. "Properties of the Polynomials Defined by Morgan–Voyce." *The Fibonacci Quarterly* **4.1** (1966):73-81.
10. M. N. S. Swamy. "Further Properties of Morgan–Voyce Polynomials." *The Fibonacci Quarterly* **6.2** (1968):167-75.

AMS Classification Numbers: 11B39, 26A24, 11B83

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