# DIFFERENTIAL PROPERTIES OF A GENERAL CLASS OF POLYNOMIALS 

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## 1. INTRODUCTION

Let us consider the generalized Fibonacci polynomials $U_{n}(p, q ; x)$ and the generalized Lucas polynomials $V_{n}(p, q ; x)$ (or simply $U_{n}$ and $V_{n}$ if there is no danger of confusion) defined by

$$
\begin{equation*}
U_{n}=(x+p) U_{n-1}-q U_{n-2} \quad\left(U_{0}=0, U_{1}=1\right), \tag{1.1}
\end{equation*}
$$

and

$$
\begin{equation*}
V_{n}=(x+p) V_{n-1}-q V_{n-2} \quad\left(V_{0}=2, V_{1}=x+p\right) . \tag{1.2}
\end{equation*}
$$

The parameters $p$ and $q$ as well as the variable $x$ are arbitrary real numbers and we denote by $\alpha=\alpha(x)$ and $\beta=\beta(x)$ the numbers such that $\alpha+\beta=x+p$ and $\alpha \beta=q$. The polynomials $U_{n}$ and $V_{n}$ can be expressed by means of the Binet forms

$$
\begin{equation*}
U_{n}=\frac{\alpha^{n}-\beta^{n}}{\Delta^{1 / 2}}, \text { for } \Delta \neq 0 \tag{1.3}
\end{equation*}
$$

and

$$
\begin{equation*}
V_{n}=\alpha^{n}+\beta^{n}, \tag{1.4}
\end{equation*}
$$

where

$$
\begin{equation*}
\Delta=\Delta(x)=(x+p)^{2}-4 q . \tag{1.5}
\end{equation*}
$$

Recall that

$$
\begin{equation*}
\alpha=\left((x+p)+\Delta^{1 / 2}\right) / 2, \quad \beta=\left((x+p)-\Delta^{1 / 2}\right) / 2 . \tag{1.6}
\end{equation*}
$$

Notice that $\Delta>0$ for every $x$ if $q<0$ for all $x$ sufficiently large if $q \geq 0$.
Particular cases of $U_{n}(p, q ; x)$ and $V_{n}(p, q ; x)$ are the Fibonacci and Lucas polynomials ( $F_{n}(x)$ and $L_{n}(x)$ ), the Pell and Pell-Lucas polynomials [6] $\left(P_{n}(x)\right.$ and $\left.Q_{n}(x)\right)$, the first and the second Fermat polynomials [7] $\left(\Phi_{n}(x)\right.$ and $\left.\Theta_{n}(x)\right)$, the Morgan-Voyce polynomials [1, 2, 5, 8, 9, 10] ( $B_{n}(x)$ and $C_{n}(x)$ ), and the Chebyschev polynomials $\left(S_{n}(x)\right.$ and $\left.T_{n}(x)\right)$ given by

$$
\begin{array}{ll}
U_{n}(0,-1 ; x)=F_{n}(x), & V_{n}(0,-1 ; x)=L_{n}(x) \\
U_{n}(0,-1 ; 2 x)=P_{n}(x), & V_{n}(0,-1 ; 2 x)=Q_{n}(x) \\
U_{n}(0,2 ; x)=\Phi_{n}(x), & V_{n}(0,2 ; x)=\Theta_{n}(x)  \tag{1.7}\\
U_{n+1}(2,1 ; x)=B_{n}(x), & V_{n}(2,1 ; x)=C_{n}(x) \\
U_{n}(0,1 ; 2 x)=S_{n}(x), & V_{n}(0,1 ; 2 x)=2 T_{n}(x) .
\end{array}
$$

In earlier papers $[1,2]$ the author has discussed the combinatorial properties of the coefficients of $U_{n}$ and $V_{n}$. Here, we shall investigate the differential properties satisfied by these polynomials, such as differential equations and Rodrigues' formulas.

Let us define the sequence $\left\{c_{n, k}\right\}_{n \geq k \geq 0}$ by

$$
\begin{equation*}
c_{n, 0}=2 \frac{n!}{(2 n)!}, \quad n \geq 0 \tag{1.8}
\end{equation*}
$$

and

$$
\begin{equation*}
c_{n, k}=2 \frac{n!}{(2 n)!} \frac{n}{n+k} \frac{(n+k)!}{(n-k)!}, \quad n \geq k \geq 1 . \tag{1.9}
\end{equation*}
$$

Notice that

$$
\begin{equation*}
c_{n, k+1}=\left(n^{2}-k^{2}\right) c_{n, k}, \quad n \geq k+1 \geq 1 . \tag{1.10}
\end{equation*}
$$

Our main results are the following theorems.
Theorem 1: For every real number $x$, the polynomial

$$
U_{n}^{(k-1)}=\frac{d^{k-1}}{d x^{k-1}} U_{n}, \quad k \geq 1,
$$

and the polynomial

$$
V_{n}^{(k)}=\frac{d^{k}}{d x^{k}} V_{n}, \quad k \geq 0
$$

satisfy the differential equation $E_{n, k}$ :

$$
\begin{equation*}
\Delta z^{\prime \prime}+(2 k+1)(x+p) z^{\prime}+\left(k^{2}-n^{2}\right) z=0 . \tag{1.11}
\end{equation*}
$$

Theorem 2: For every $x$ such that $\Delta>0$, we have

$$
\begin{equation*}
U_{n}=n c_{n, 0} \Delta^{-1 / 2} \frac{d^{n-1}}{d x^{n-1}} \Delta^{n-1 / 2}, \quad n \geq 1 \tag{1.12}
\end{equation*}
$$

and

$$
\begin{equation*}
V_{n}=c_{n, 0} \Delta^{1 / 2} \frac{d^{n}}{d x^{n}} \Delta^{n-1 / 2}, \quad n \geq 0, \tag{1.13}
\end{equation*}
$$

where $c_{n, 0}$ is defined by (1.8).
More generally, we also have Rodrigues' formulas for $U_{n}^{(k)}$ and $V_{n}^{(k)}$, namely,
Theorem 3: For every $x$ such that $\Delta>0$ and every $k \geq 0$, we have

$$
\begin{equation*}
U_{n}^{(k)}=\frac{\left(n^{2}-k^{2}\right)}{n} c_{n, k} \Delta^{-k-1 / 2} \frac{d^{n-k-1}}{d x^{n-k-1}} \Delta^{n-1 / 2}, \quad n \geq k+1, \tag{1.14}
\end{equation*}
$$

and

$$
\begin{equation*}
V_{n}^{(k)}=c_{n, k} \Delta^{-k+1 / 2} \frac{d^{n-k}}{d x^{n-k}} \Delta^{n-1 / 2}, \quad n \geq k \tag{1.15}
\end{equation*}
$$

where $c_{n, k}$ is defined by (1.9).
Notice that Theorem 3 reduces to Theorem 2 for $k=0$ and that (1.14) can be written, by (1.10),

$$
\begin{equation*}
U_{n}^{(k)}=\frac{c_{n, k+1}}{n} \Delta^{-k-1 / 2} \frac{d^{n-k-1}}{d x^{n-k-1}} \Delta^{n-1 / 2}, \quad n \geq k . \tag{1.16}
\end{equation*}
$$

## 2. PROOF OF THEOREM 1

It is readily proven $[3,4]$ by (1.5) and (1.6) that, for every $x$ such that $\Delta>0$,

$$
\left\{\begin{array}{l}
\alpha^{\prime}=\alpha \Delta^{-1 / 2}  \tag{2.1}\\
\beta^{\prime}=-\beta \Delta^{-1 / 2}
\end{array}\right.
$$

and thus that

$$
\left\{\begin{array}{l}
\left(\alpha^{n}\right)^{\prime}=n \alpha^{n} \Delta^{-1 / 2},  \tag{2.2}\\
\left(\beta^{n}\right)^{\prime}=-n \beta^{n} \Delta^{-1 / 2}
\end{array}\right.
$$

By this, (1.3), and (1.4), we see [3, 4] that

$$
\begin{equation*}
V_{n}^{\prime}=n U_{n} \tag{2.3}
\end{equation*}
$$

and therefore that

$$
\begin{equation*}
V_{n}^{(k)}=n U_{n}^{(k-1)}, \quad k \geq 1 . \tag{2.4}
\end{equation*}
$$

Notice that these identities are valid for every value of $x$, and not only when $\Delta>0$, since the two members are polynomials. By (2.2), we also deduce that $\alpha^{n}$ and $\beta^{n}$, whence $V_{n}=\alpha^{n}+\beta^{n}$ satisfies the differential equation

$$
\begin{equation*}
\frac{d}{d x}\left(\Delta^{1 / 2} y^{\prime}\right)=n^{2} \Delta^{-1 / 2} y, \text { for } \Delta>0 \tag{2.5}
\end{equation*}
$$

which is equivalent, for $\Delta>0$, to the equation $E_{n, 0}$ [see (1.11)], namely,

$$
\begin{equation*}
\Delta y^{\prime \prime}+(x+p) y^{\prime}-n^{2} y=0 \tag{2.6}
\end{equation*}
$$

Notice that $V_{n}$ satisfies $E_{n, 0}$ for every value of $x$, since, in that case, the first member of (2.6) is a polynomial.

Differentiating (2.6) $k$ times and using Leibniz' rule, we see that $z=y^{(k)}$ satisfies the differential equation $E_{n, k}$ (1.11). Hence, $E_{n, k}$ is satisfied by $V_{n}^{(k)}, k \geq 0$, and $U_{n}^{(k-1)}=\frac{1}{n} V_{n}^{(k)}, k \geq 1$. This concludes the proof.

For instance, the Morgan-Voyce polynomial $B_{n}(x)=U_{n+1}(2,1 ; x)$ satisfies the differential equation $E_{n+1,1}$

$$
x(x+4) z^{\prime \prime}+3(x+2) z^{\prime}-n(n+2) z=0 .
$$

This result was first noticed by Swamy [10].
Remark: When $\Delta>0$, it is easy to verify that $E_{n, k}$ can be written as

$$
\begin{equation*}
\frac{d}{d x}\left[\Delta^{k+1 / 2} z^{\prime}\right]=\left(n^{2}-k^{2}\right) \Delta^{k-1 / 2} z \tag{2.7}
\end{equation*}
$$

which is a generalization of (2.5).
We now give another (nonpolynomial) solution of $E_{n, k}$.
Proposition 1: Let $n$ and $k$ be two integers such that $n+k-1 \geq 0$. Then, for $\Delta>0$, the function $\frac{d^{n+k-1}}{d x^{n+k-1}} \Delta^{n-1 / 2}$ is a solution of $E_{n, k}$.

Proof: It is easy to verify that, for $\Delta>0, \Delta^{n-1 / 2}$ is a solution of the differential equation

$$
\begin{equation*}
\Delta y^{\prime \prime}-(2 n-3)(x+p) y^{\prime}-(2 n-1) y=0 . \tag{2.8}
\end{equation*}
$$

Differentiating (2.8) $(n+k-1)$ times and putting $z=y^{(n+k-1)}$, we obtain

$$
\Delta z^{\prime \prime}+2\binom{n+k-1}{1}(x+p) z^{\prime}+2\binom{n+k-1}{2} z-(2 n-3)\left[(x+p) z^{\prime}+\binom{n+k-1}{1} z\right]-(2 n-1) z=0 .
$$

After some rearrangement, one can see that this equation is identical to $E_{n, k}$.
Remark: Using the formulation (2.7) of $E_{n, k}$ and putting $z=\frac{d^{n+k-1}}{d x^{n+k-1}} \Delta^{n-1 / 2}$, one can write

$$
\begin{equation*}
\frac{d}{d x}\left[\Delta^{k+1 / 2} \frac{d^{n+k}}{d x^{n+k}} \Delta^{n-1 / 2}\right]=\left(n^{2}-k^{2}\right) \Delta^{k-1 / 2} \frac{d^{n+k-1}}{d x^{n+k-1}} \Delta^{n-1 / 2} \tag{2.9}
\end{equation*}
$$

Changing $k$ to $(-k-1)$ in (2.9), where $n-k \geq 2$, we obtain a formula that we shall need later:

$$
\begin{equation*}
\frac{d}{d x}\left[\Delta^{-k-1 / 2} \frac{d^{n-k-1}}{d x^{n-k-1}} \Delta^{n-1 / 2}\right]=\left(n^{2}-(k+1)^{2}\right) \Delta^{-k-3 / 2} \frac{d^{n-k-2}}{d x^{n-k-2}} \Delta^{n-1 / 2} . \tag{2.10}
\end{equation*}
$$

In particular, changing $n$ to $(n+1)$, and putting $k=-1$, we get

$$
\begin{equation*}
\frac{d}{d x}\left[\Delta^{1 / 2} \frac{d^{n+1}}{d x^{n+1}} \Delta^{n+1 / 2}\right]=(n+1)^{2} \Delta^{-1 / 2} \frac{d^{n}}{d x^{n}} \Delta^{n+1 / 2}, n \geq 0 . \tag{2.11}
\end{equation*}
$$

## 3. PROOF OF THEOREM 2

In the proof of Theorem 2, we shall need the following well-known and readily proven result:

$$
\begin{equation*}
V_{n+1}=\frac{1}{2}\left[(x+p) V_{n}+\Delta U_{n}\right] . \tag{3.1}
\end{equation*}
$$

By (1.8), formula (1.12) (resp. (1.13)) is clear if $n=1$ (resp. $n=0$ or $n=1$ ). Supposing that (1.12) and (1.13) are true for $n \geq 1$, we get by (3.1) that

$$
\begin{equation*}
V_{n+1}=\frac{n!}{(2 n!)} \Delta^{1 / 2}\left[(x+p) \frac{d^{n}}{d x^{n}} \Delta^{n-1 / 2}+n \frac{d^{n-1}}{d x^{n-1}} \Delta^{n-1 / 2}\right] . \tag{3.2}
\end{equation*}
$$

On the other hand, one can notice by (1.5) that

$$
\begin{align*}
\frac{d^{n+1}}{d x^{n+1}} \Delta^{n+1 / 2} & =\frac{d^{n}}{d x^{n}}\left[(2 n+1)(x+p) \Delta^{n-1 / 2}\right] \\
& =(2 n+1)\left[(x+p) \frac{d^{n}}{d x^{n}} \Delta^{n-1 / 2}+n \frac{d^{n-1}}{d x^{n-1}} \Delta^{n-1 / 2}\right] . \tag{3.3}
\end{align*}
$$

From (3.2) and (3.3), we see that

$$
\begin{equation*}
V_{n+1}=\frac{n!}{(2 n!)} \Delta^{1 / 2} \frac{1}{2 n+1} \frac{d^{n+1}}{d x^{n+1}} \Delta^{n+1 / 2}=2 \frac{(n+1)!}{(2 n+2)!} \Delta^{1 / 2} \frac{d^{n+1}}{d x^{n+1}} \Delta^{n+1 / 2}, \tag{3.4}
\end{equation*}
$$

which is the needed formula for $V_{n+1}$.

Now we see, by (2.3) and (3.4), that

$$
\begin{align*}
U_{n+1} & =\frac{1}{n+1} V_{n+1}^{\prime}=2 \frac{n!}{(2 n+2)!} \frac{d}{d x}\left[\Delta^{1 / 2} \frac{d^{n+1}}{d x^{n+1}} \Delta^{n+1 / 2}\right] \\
& =2 \frac{n!}{(2 n+2)!}(n+1)^{2} \Delta^{-1 / 2} \frac{d^{n}}{d x^{n}} \Delta^{n+1 / 2}, \quad \text { by }(2.11),  \tag{3.5}\\
& =2(n+1) \frac{(n+1)!}{(2 n+2)!} \Delta^{-1 / 2} \frac{d^{n}}{d x^{n}} \Delta^{n+1 / 2}
\end{align*}
$$

This completes the proof of Theorem 2.

## 4. PROOF OF THEOREM 3

We proceed by induction on $k$. By Theorem 2, statement (1.14) clearly holds for $k=0$ and every $n \geq 1$. Supposing that (1.14) holds for $k \geq 0$ and every $n \geq k+1$ we get, by (1.16),

$$
U_{n}^{(k+1)}=\frac{d}{d x} U_{n}^{(k)}=\frac{c_{n, k+1}}{n} \frac{d}{d x}\left[\Delta^{-k-1 / 2} \frac{d^{n-k-1}}{d x^{n-k-1}} \Delta^{n-1 / 2}\right],
$$

and, by (2.10), we have at once that

$$
U_{n}^{(k+1)}=\frac{c_{n, k+1}}{n}\left[n^{2}-(k+1)^{2}\right] \Delta^{-k-3 / 2} \frac{d^{n-k-2}}{d x^{n-k-2}} \Delta^{n-1 / 2}, \quad n \geq k+2,
$$

which is the needed formula for $U_{n}^{(k+1)}$.
On the other hand, statement (1.15) holds for $k=0$, by Theorem 2 . When $k \geq 1$ we get, by (2.4) and (1.14) that

$$
V_{n}^{(k)}=n U_{n}^{(k-1)}=c_{n, k} \Delta^{-k+1 / 2} \frac{d^{n-k}}{d x^{n-k}} \Delta^{n-1 / 2}, \quad n \geq k .
$$

This completes the proof of Theorem 3.

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