ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
Stanley Rabinowitz

Please send all material for ELEMENTARY PROBLEMS AND SOLUTIONS to Dr. STANLEY RABINOWITZ; 12 VINE BROOK RD; WESTFORD, MA 01886-4212 USA. Correspondence may also be sent to the problem editor by electronic mail to Fibonacci@MathPro.com *on Internet. All correspondence will be acknowledged.*

Each solution should be on a separate sheet (or sheets) and must be received within six months of publication of the problem. Solutions typed in the format used below will be given preference. Proposers of problems should normally include solutions. Proposers should inform us of the history of the problem, if it is not original. A problem should not be submitted elsewhere while it is under consideration for publication in this column.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$
$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-796 ***Proposed by M. N. S. Swamy, St. Lambert, Quebec, Canada***

Show that $\dfrac{L_n^2 + L_{n+1}^2 + L_{n+2}^2 + \cdots + L_{n+a}^2}{F_n^2 + F_{n+1}^2 + F_{n+2}^2 + \cdots + F_{n+a}^2}$ is always an integer if a is odd.

B797 ***Proposed by Andrew Cusumano, Great Neck, NY***

Let $\langle H_n \rangle$ be any sequence that satisfies the recurrence $H_{n+2} = H_{n+1} + H_n$. Prove that

$$7H_n \equiv H_{n+15} \pmod{10}.$$

B-798 ***Proposed by Seung-Jin Bang, Ajou University, Suwon, Korea***

Prove that, for n a positive integer, F_{5^n} is divisible by 5^n but not by 5^{n+1}.

B-799 ***Proposed by David Zeitlin, Minneapolis, MN***

Solve the recurrence $A_{n+2} = 4A_{n+1} + A_n$, for $n \geq 0$, with initial conditions $A_0 = 1$ and $A_1 = 4$; expressing your answer in terms of Fibonacci and/or Lucas numbers.

B-800 ***Proposed by H.-J. Seiffert, Berlin, Germany***

Define the Pell numbers by the recurrence $P_n = 2P_{n-1} + P_{n-2}$, for $n \geq 2$, with initial conditions $P_0 = 0$ and $P_1 = 1$.

Show that, for all integers $n \geq 4$, $P_n < F_{k(n)}$ where $k(n) = \lfloor (11n+2)/6 \rfloor$.

B-801 ***Proposed by Larry Taylor, Rego Park, NY***

Let $k \geq 2$ be an integer and let n be an odd integer. Prove that

(a) $$F_{2^k} \equiv 27 \cdot 7^k \pmod{40};$$

(b) $$F_{n2^k} \equiv 7^k F_{16n} \pmod{40}.$$

SOLUTIONS

A Lucas Congruence

B-766 ***Proposed by R. André-Jeannin, Longwy, France***
(Vol. 32, no. 4, August 1994)

Let n be an even positive integer such that $L_n \equiv 2 \pmod{p}$, where p is an odd prime. Prove that $L_{n+1} \equiv 1 \pmod{p}$.

Solution by Leonard A. G. Dresel, Reading, England

We start with the identity
$$5F_n^2 = L_n^2 - 4(-1)^n,$$
which is identity (24) from [1]. If n is even and $L_n \equiv 2 \pmod{p}$, we have $5F_n^2 \equiv 0 \pmod{p}$ and therefore $5F_n \equiv 0 \pmod{p}$ since p is a prime. Applying the identity $L_{n+1} + L_{n-1} = 5F_n$, which is identity (5) from [1], and the definition $L_{n+1} - L_{n-1} = L_n$, we find $2L_{n+1} = 5F_n + L_n \equiv 2 \pmod{p}$. Since p is odd, this gives $L_{n+1} \equiv 1 \pmod{p}$.

Reference:

1. S. Vajda. *Fibonacci & Lucas Numbers, and the Golden Section: Theory and Applications.* Chichester, England: Ellis Horwood Ltd., 1989.

Also solved by Paul S. Bruckman, Herta T. Freitag, Norbert Jensen, Bob Prielipp, H.-J. Seiffert, Lawrence Somer, David C. Terr, and the proposer.

Mutual Admiration Fibonacci Society

B-767 ***Proposed by James L. Hein, Portland State University, Portland, OR***
(Vol. 32, no. 4, August 1994)

Consider the following two mutual recurrences:
$$G_1 = 1; \quad G_n = F_{n+1}G_{n-1} + F_n H_{n-2}, \ n \geq 2;$$
and
$$H_0 = 0; \quad H_n = F_{n+1}G_n + F_n H_{n-1}, \ n \geq 1.$$

Prove that H_{n-1} and G_n are consecutive Fibonacci numbers for all $n \geq 1$.

Solution by M. N. S. Swamy, Montreal, Canada

We see that G_n and H_{n-1} are consecutive Fibonacci numbers for $n = 1$ and $n = 2$ since $G_1 = F_1$, $H_0 = F_0$, and $G_2 = F_3$, $H_1 = F_2$. Assuming that $G_n = F_{a_n}$ and $H_{n-1} = F_{a_n - 1}$, where $a_n = n(n+1)/2$, we have
$$G_{n+1} = F_{n+2}F_{a_n} + F_{n+1}F_{a_n - 1} = F_{n+1+a_n} = F_{a_{n+1}},$$
where we have used identity (I_{26}) from [1]: $F_{j+1}F_{k+1} + F_j F_k = F_{j+k+1}$. In the same way,

$$H_n = F_{n+1}F_{a_n} + F_nF_{a_n-1} = F_{n+a_n} = F_{a_{n+1}-1}.$$

Hence, by induction, we have $G_n = F_{a_n}$ and $H_{n-1} = F_{a_n-1}$ for all n. Thus, G_n and H_{n-1} are consecutive Fibonacci numbers for all $n \geq 1$.

Reference:

1. Verner E. Hoggatt, Jr. *Fibonacci and Lucas Numbers.* Santa Clara, CA: The Fibonacci Association, 1979.

Also solved by Paul S. Bruckman, Charles K. Cook, Leonard A. G. Dresel, Steve Edwards, Herta T. Freitag, C. Georghiou, Norbert Jensen, Carl Libis, Bob Prielipp, Don Redmond, H.-J. Seiffert, Lawrence Somer, David Zeitlin, and the proposer.

A Radical Approach to Fibonacci Numbers

B-768 ***Proposed by Juan Pla, Paris, France***
(Vol. 32, no. 4, August 1994)

Let u_n, v_n, and w_n be sequences defined by $u_1 = 1/2, v_1 = \sqrt{2}$, and $w_1 = (1/2)\sqrt{3}$; $u_{n+1} = u_n^2 + v_n^2 - w_n^2$, $v_{n+1} = 2u_nv_n$, $w_{n+1} = 2u_nw_n$. Express u_n, v_n, and w_n in terms of Fibonacci and/or Lucas numbers.

Solution by C. Georghiou, University of Patras, Greece

The answer is $u_n = \frac{1}{2}L_m$, $v_n = \sqrt{2}F_m$, and $w_n = \frac{1}{2}\sqrt{3}F_m$, where $m = 2^{n-1}$. We prove this by induction. Evidently, it is true for $n = 1$. Assuming it is true for n, we have

$$u_{n+1} = \tfrac{1}{4}L_m^2 + 2F_m^2 - \tfrac{3}{4}F_m^2 = \tfrac{1}{4}(L_m^2 + 5F_m^2) = \tfrac{1}{2}L_{2m} = \tfrac{1}{2}L_{2^n},$$

where we have used the identity $L_m^2 + 5F_m^2 = 2L_{2m}$, which is identity (22) from [1]. We also have

$$v_{n+1} = \sqrt{2}L_mF_m = \sqrt{2}F_{2m} = \sqrt{2}F_{2^n}$$

and

$$w_{n+1} = \tfrac{1}{2}\sqrt{3}L_mF_m = \tfrac{1}{2}\sqrt{3}F_{2m} = \tfrac{1}{2}\sqrt{3}F_{2^n},$$

where we have used the identity $L_mF_m = F_{2m}$, which is identity (13) from [1]. The induction step is now complete.

Reference:

1. S. Vajda. *Fibonacci & Lucas Numbers, and the Golden Section: Theory and Applications.* Chichester, England: Ellis Horwood Ltd., 1989.

Also solved by Brian D. Beasley, Paul S. Bruckman, Charles K. Cook, Leonard A. G. Dresel, Steve Edwards, Herta T. Freitag, Norbert Jensen, Hans Kappus, Bob Prielipp, H.-J. Seiffert, David C. Terr, David Zeitlin, and the proposer.

The Recurrence for F_{3^n}

B-769 ***Proposed by Piero Filipponi, Fond. U. Bordoni, Rome, Italy***
(Vol. 32, no. 4, August 1994)

Solve the recurrence $a_{n+1} = 5a_n^3 - 3a_n$, $n \geq 0$, with initial condition $a_0 = 1$.

Solution by David C. Terr, University of California, Berkeley

We claim that the solution is $a_n = F_{3^n}$. Clearly this holds for $n = 0$. Assume it holds for some nonnegative integer n. Then

$$\begin{aligned} a_{n+1} &= 5a_n^3 - 3a_n = 5F_{3^n}^3 - 3F_{3^n} \\ &= 5\left[\frac{1}{\sqrt{5}^3}(\alpha^{3^n} - \beta^{3^n})^3\right] - \frac{3}{\sqrt{5}}(\alpha^{3^n} - \beta^{3^n}) \\ &= \frac{1}{\sqrt{5}}(\alpha^{3^{n+1}} - \beta^{3^{n+1}} - 3(\alpha^{3^n} - \beta^{3^n})[(\alpha\beta)^{3^n} + 1]) \\ &= \frac{1}{\sqrt{5}}(\alpha^{3^{n+1}} - \beta^{3^{n+1}}) = F_{3^{n+1}}, \end{aligned}$$

where we have used the identity $\alpha\beta = -1$. Thus, by induction, our answer is correct for all nonnegative integers n.

Comment by Murray S. Klamkin, University of Alberta, Canada

The same problem appeared as Problem 1809 in *Crux Mathematicorum* **20** (1994):19-20. In the same issue, there was a proposal to solve the recurrence

$$P_{n+1} = 25P_n^5 - 25P_n^3 + 5P_n, \quad P_0 = 1.$$

The solution, which appeared in **20** (1994):295-96, is $P_n = F_{5^n}$. Also, one can show that the solutions to the following recurrences

$$\begin{aligned} A_{n+1} &= A_n^2 - 2, & A_1 &= 3, \\ B_{n+1} &= B_n^4 - 4B_n^2 + 2, & B_1 &= 7, \\ C_{n+1} &= C_n^6 - 6C_n^4 + 9C_n^2 - 2, & C_1 &= 18, \end{aligned}$$

are given by $A_n = L_{2^n}$, $B_n = L_{4^n}$, and $C_n = L_{6^n}$.

In the Crux Mathematicorum solution, it was shown that the solution to the recurrence $p_0 = 1$, $p_{n+1} = \frac{1}{\sqrt{5}} f_m(\sqrt{5}p_n)$, *m odd*, $m \geq 3$, *where* $f(x)$ *is defined by* $f_0(x) = 2$, $f_1(x) = x$, *and* $f_n(x) = xf_{n-1}(x) - f_{n-2}(x)$, *for* $n \geq 2$ *is* $p_n = F_{m^n}$. *This reduces to our problem when* $m = 3$.

Also solved by Michel A. Ballieu, Seung-Jin Bang, Brian D. Beasley, Paul S. Bruckman, Leonard A. G. Dresel, Steve Edwards, Herta T. Freitag, C. Georghiou, Norbert Jensen, Hans Kappus, Murray S. Klamkin, Bob Prielipp, H.-J. Seiffert, Lawrence Somer, Adam Stinchcombe, David Zeitlin, and the proposer.

Unit Digit Madness

B-770 ***Proposed by Andrew Cusumano, Great Neck, NY***
(Vol. 32, no. 4, August 1994)

Let $U(x)$ denote the unit's digit of x when written in base 10. Let H_n be any generalized Fibonacci sequence that satisfies the recurrence $H_n = H_{n-1} + H_{n-2}$. Prove that, for all n,

$$\begin{array}{ll}
U(H_n+H_{n+4})=U(H_{n+47}), & U(H_n+H_{n+17})=U(H_{n+34}),\\
U(H_n+H_{n+5})=U(H_{n+10}), & U(H_n+H_{n+19})=U(H_{n+41}),\\
U(H_n+H_{n+7})=U(H_{n+53}), & U(H_n+H_{n+20})=U(H_{n+55}),\\
U(H_n+H_{n+8})=U(H_{n+19}), & U(H_n+H_{n+23})=U(H_{n+37}),\\
U(H_n+H_{n+11})=U(H_{n+49}), & U(H_n+H_{n+25})=U(H_{n+50}),\\
U(H_n+H_{n+13})=U(H_{n+26}), & U(H_n+H_{n+28})=U(H_{n+59}),\\
U(H_n+H_{n+16})=U(H_{n+23}), & U(H_n+H_{n+29})=U(H_{n+58}).
\end{array}$$

Solution by Paul S. Bruckman, Edmonds, WA

Essentially, the problem asks us to verify that $H_n+H_{n+a}\equiv H_{n+b}$ (mod 10), for all n, where (a,b) is a specified pair of positive integers. Using the identity

$$H_n=F_nH_1+F_{n-1}H_0,$$

which is identity (8) of [1], we see that it suffices to prove that

$$F_n+F_{n+a}\equiv F_{n+b}\ (\text{mod}\,10),\quad \text{for all } n. \tag{$*$}$$

Since $F_m+F_{m+1}=F_{m+2}$, we need only prove $(*)$ for $n=0$ and $n=1$, for then, by induction, $(*)$ would be true for all n. Thus, we need only show that $U(F_a)=U(F_b)$ and $U(1+F_{a+1})=U(F_{b+1})$ for the given a and b.

In each case, these are readily checked from the following table of $U(F_n), n=1,2,\ldots,60$:

1, 1, 2, 3, 5, 8, 3, 1, 4, 5, 9, 4, 3, 7, 0, 7, 7, 4, 1, 5, 6, 1, 7, 8, 5, 3, 8, 1, 9, 0,
9, 9, 8, 7, 5, 2, 7, 9, 6, 5, 1, 6, 7, 3, 0, 3, 3, 6, 9, 5, 4, 9, 3, 2, 5, 7, 2, 9, 1, 0.

Reference:

1. S. Vajda. *Fibonacci & Lucas Numbers, and the Golden Section: Theory and Applications.* Chichester, England: Ellis Horwood Ltd., 1989.

Also solved by Leonard A. G. Dresel, Herta T. Freitag, Norbert Jensen, Bob Prielipp, H.-J. Seiffert, David Zeitlin, and voluminous generalizations and correspondence by the proposer.

More Sums

B-771 ***Proposed by H.-J. Seiffert, Berlin, Germany***
(Vol. 32, no. 4, August 1994)

Show that

$$\sum_{n=1}^{\infty}\frac{(2n+1)F_n}{2^n n(n+1)}=\ln 4.$$

Solution by Don Redmond, Southern Illinois University, Carbondale, IL

We generalize this result somewhat.

Let $r, t,$ and u be complex numbers such that $|t/r|<1$ and $|u/r|<1$. Define the sequence $\langle P_n\rangle$ by $P_n=ct^n+du^n$, where c and d are arbitrary complex numbers. For $|x|<1$, we know that

$$\sum_{n=1}^{\infty}\frac{x^n}{n}=\ln\left(\frac{1}{1-x}\right).$$

This is series 1.513 on page 44 of [1]. Since

$$\frac{2n+1}{n(n+1)}=\frac{1}{n}+\frac{1}{n+1},$$

we find that

$$\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}x^n=-1-\left(1+\frac{1}{x}\right)\ln(1-x).$$

Thus,

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{2n+1}{r^n n(n+1)}P_n&=\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}\left[c\left(\frac{t}{r}\right)^n+d\left(\frac{u}{r}\right)^n\right]\\&=c\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}\left(\frac{t}{r}\right)^n+d\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}\left(\frac{u}{r}\right)^n\\&=-c\left(1+\frac{r}{t}\right)\ln\left(1-\frac{t}{r}\right)-c-d\left(1+\frac{r}{u}\right)\ln\left(1-\frac{u}{r}\right)-d.\end{aligned}$$

If $t=\alpha,\ u=\beta,\ c=1/\sqrt{5},\ d=-1/\sqrt{5}$, and $r=2$, we get

$$\sum_{n=1}^{\infty}\frac{(2n+1)}{2^n n(n+1)}F_n=\ln 4.$$

If $t=\alpha,\ u=\beta,\ c=1,\ d=1$, and $r=2$, we get

$$\sum_{n=1}^{\infty}\frac{(2n+1)}{2^n n(n+1)}L_n=-2-\sqrt{5}\ln\left(\frac{7-3\sqrt{5}}{2}\right).$$

Reference:

1. I. S. Gradshteyn & I. M. Ryzhik. *Table of Integrals, Series, and Products.* San Diego, CA: Academic Press, 1980.

Also solved by Seung-Jin Bang, Glenn A. Bookhout, Wray Brady, Paul S. Bruckman, Leonard A. G. Dresel, Steve Edwards, C. Georghiou, Norbert Jensen, Hans Kappus, Murray S. Klamkin, Bob Prielipp, Adam Stinchcombe, David Zeitlin, and the proposer.

ERRATA

B-746 (Feb. 1995, p. 87): It should be noted that the formula $L_{3n}=L_n^3+3L_n$ is only valid for n odd.

B-754 (May 1995): Gauthier's formula on the bottom of page 184 should read

$$\sum_{k=1}^{n}x^kG_{sk+t}=\frac{(-1)^s x^{n+1}G_{sn+t}-x^nG_{x(n+1)+t}+G_{s+t}+(-1)^{s+1}xG_t}{1-2x(G_s+G_{s-1})+(-1)^s x^2}.$$

B-759 (Aug. 1995, p. 372): In the fourth line of the solution, $tu^{n+1}(t/v)^j$ should be $tv^{n+1}(t/v)^j$.

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