THE SWITCH, SUBTRACT, REORDER ROUTINE

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Let g be a positive integer greater than 1. An integer x is an **ordered 4-digit base g number** if

$$x = a_1 \cdot g^3 + a_2 \cdot g^2 + a_3 \cdot g + a_4 = a_1\, a_2\, a_3\, a_4 \text{ base } g$$

with

$$g > a_1 \geq a_2 \geq a_3 \geq a_4 \geq 0.$$

The following procedure, when applied to x, yields another ordered 4-digit base g number. Switch the first two digits of x with the last two digits; then subtract the "switched" number from x:

$$a_1\, a_2\, a_3\, a_4 - a_3\, a_4\, a_1\, a_2 = b_1\, b_2\, b_3\, b_4 \text{ base } g.$$

Finally, rearrange the digits so that the result, $R(x)$, is an ordered number. Thus,

$$R(x) = a_1'\, a_2'\, a_3'\, a_4' \text{ base } g,$$

where the a_i''s are a permutation of the b_i's and $a_i' \geq a_2' \geq a_3' \geq a_4'$. This procedure is called the **Switch, Subtract, Reorder (SSR) routine**.

As an example of the SSR routine, consider the base 15 number $x = 13\ 12\ 10\ 8$. We switch the digits of x and subtract:

$$13\ 12\ 10\ 8 - 10\ 8\ 13\ 12 = 3\ 3\ 11\ 11_{\text{base } 15}.$$

When we reorder, we get $R(x) = 11\ 11\ 3\ 3$.

Returning to the general case, we can apply the SSR routine to the number $R(x)$ to get $R(R(x)) = R^2(x)$. More generally, $R^{i+1}(x) = R(R^i(x))$ for $i \geq 1$. Since there are only finitely many ordered 4-digit base g numbers, repetition must occur. That is, $R^j(x) = R^i(x)$ for some i and j.

Continuing to use base 15 as an example, we will find the iterates of $x = 13\ 12\ 10\ 8$. For reasons which will become clear shortly, for each ordered number $a_1\, a_2\, a_3\, a_4$, we will also calculate the difference $diff = a_2 - a_4$. The results are given in the following table:

TABLE 1

i	$R^i(x)$	*diff*
1	11 11 3 3	8
2	8 7 7 6	1
3	14 13 1 0	13
4	13 12 2 1	11
5	11 10 4 3	7
6	8 7 7 6	1

Notice that $R^2(x)=R^6(x)$. We write $\langle R^2(x), R^3(x), R^4(x), R^5(x)\rangle$ and call this expression a **cycle** of length 4. We say that $R^2(x)$ **generates** the cycle. Of course, $R^3(x), R^4(x)$, and $R^5(x)$ also generate the cycle.

As the example above illustrates, for a given g the SSR routine always gives rise to at least one cycle. We will characterize the cycles by answering the following questions: How many are there? What number(s) is(are) in each cycle? What is the length of each cycle? What is the smallest i such that, for all x, $R^i(x)$ is in a cycle?

The SSR routine is a variation of the Kaprekar routine. For the Kaprekar routine, we reverse the digits of x and subtract:

$$a_1\, a_2\, a_3\, a_4 - a_4\, a_3\, a_2\, a_1 = b_1\, b_2\, b_3\, b_4 \text{ base } g.$$

Reordering the digits gives $K(x)$. The Kaprekar routine has been studied extensively (see [1]-[8]). Among the questions addressed are those that appear at the end of the previous paragraph.

A REDUCTION OF THE PROBLEM

For an ordered number $x=a_1\, a_2\, a_3\, a_4$, the **digit differences** of x are $D=a_1-a_3$ and $d=a_2-a_4$. Clearly, $0\le D,\ d<g$. Now, if $R(x)=a_1'\, a_2'\, a_3'\, a_4'$, then the digit differences of $R(x)$ are $D'=a_1'-a_3'$ and $d'=a_2'-a_4'$. As a matter of terminology, if it should happen that $D=d$, then we will refer to this number as the **digit difference** of x. That is, we will use the singular form of the noun.

The reader may wonder why we would want to look at digit differences. In part, we do so because similar digit differences play an important role in the Kaprekar routine analysis. More importantly, for a given x, $R(x)$ is completely determined by D and d. Moreover, as we now show, $D'=d'$. Thus, after one application of the SSR routine, the digit differences are equal. This observation will greatly simplify the problem of characterizing the SSR cycles.

Theorem 1: Let $x=a_1\, a_2\, a_3\, a_4$ be an ordered 4-digit base g number with digit differences D and d. Denote the digit differences of $R(x)$ by $D'=d'$. Then

$$\begin{array}{ll}
D'=d'=0 & \text{if } D=d=0,\\
D'=d'=g-D & \text{if } 0<D\le(g+1)/2\ \&\ d=0,\\
D'=d'=D-1 & \text{if } (g+1)/2\le D\ \&\ d=0,\\
D'=d'=|g-D-d| & \text{if } 0\le D\le(g-1)/2\ \&\ 0<d\le(g+1)/2\\
 & \text{or } (g-1)/2\le D\ \&\ (g+1)/2\le d,\\
D'=d'=|D-d+1| & \text{if } 0\le D\le(g-1)/2\ \&\ (g+1)/2\le d\\
 & \text{or } (g-1)/2\le D\ \&\ 0<d\le(g+1)/2.
\end{array}$$

Proof: We consider three cases.

Case 1. Suppose $D=0$ and $d=0$. Then all the digits of x are equal. In that case, $R(x)=0$ and $D'=d'=0$.

Case 2. Suppose $D\ne 0$ and $d=0$. Since $d=0$, $a_2=a_3=a_4$. We begin by switching and subtracting:

$$\begin{array}{rcccc}
 & a_1 & a_3 & a_3 & a_3\\
- & a_3 & a_3 & a_1 & a_3\\
\hline
 & D-1 & g-1 & g-D & 0
\end{array} \tag{1}$$

To find $R(x)$, we reorder the digits in (1). There are two possibilities:

$$\begin{array}{cccc} g-1 & g-D & D-1 & 0 \\ g-1 & D-1 & g-D & 0 \end{array}$$

The first occurs when $0 < D \leq (g+1)/2$; in that case, $D' = d' = g-D$. The second occurs when $(g+1)/2 \leq D$; then $D' = d' = D-1$.

<u>**Case 3.**</u> Suppose $d \neq 0$. We switch and subtract:

$$\begin{array}{cccc} a_1 & a_2 & a_3 & a_4 \\ -\,a_3 & a_4 & a_1 & a_2 \\ \hline D & d-1 & g-D-1 & g-d \end{array} \tag{2}$$

To find $R(x)$, we reorder the digits in (2). Now $D \geq d-1$ iff $g-d \geq g-D-1$; also, $D \geq g-d$ iff $d-1 \geq g-D-1$. Thus, $R(x)$ equals one of the following eight ordered numbers:

$$\begin{array}{llll} g-d & g-D-1 & D & d-1 \\ g-D-1 & g-d & d-1 & D \\ D & d-1 & g-d & g-D-1 \\ d-1 & D & g-D-1 & g-d \\ d-1 & g-D-1 & D & g-d \\ g-D-1 & d-1 & g-d & D \\ D & g-d & d-1 & g-D-1 \\ g-d & D & g-D-1 & d-1 \end{array}$$

$R(x)$ equals one of the first four numbers when

$$0 \leq D \leq (g-1)/2 \;\&\; 0 < d \leq (g+1)/2$$

or

$$(g-1)/2 \leq D \;\&\; (g+1)/2 \leq d.$$

For these numbers, $D' = d' = |g-D-d|$. On the other hand, $R(x)$ equals one of the last four numbers when

$$0 \leq D \leq (g-1)/2 \;\&\; (g+1)/2 \leq d$$

or

$$(g-1)/2 \leq D \;\&\; 0 < d \leq (g+1)/2.$$

For these numbers, $D' = d' = |D-d+1|$. □

As stated above, we refer to d' as the digit difference of $R(x)$. We now derive several corollaries. Since the first two follow immediately from the proof of Theorem 1, their proofs are omitted.

Corollary 1: Let x and y be ordered 4-digit base g numbers with digit differences D_x, d_x and D_y, d_y, respectively. If $D_x = D_y$ and $d_x = d_y$, then $R(x) = R(y)$.

Corollary 2: Let x be an ordered 4-digit base g number with equal digit differences; that is, $D = d$. Then

$$\begin{array}{ll} R(x) = 0\ \ 0\ \ 0\ \ 0 & \text{if } d = 0, \\ R(x) = g-d\ \ g-d-1\ \ d\ \ d-1 & \text{if } 0 < d < g/2, \\ R(x) = g/2\ \ g/2\ \ g/2-1\ \ g/2-1 & \text{if } d = g/2, \\ R(x) = d\ \ d-1\ \ g-d\ \ g-d-1 & \text{if } g/2 < d. \end{array}$$

Corollary 3: Let x be an ordered 4-digit base g number with equal digit differences; that is, $D = d$. Denote the digit difference of $R(x)$ by d'. Then $d' = 0$ if and only if $d = 0$. Moreover, if $d \neq 0$, then

$$\begin{array}{ll} d' = |g-2d| & \text{if } d \neq g/2, \\ d' = 1 & \text{if } d = g/2. \end{array}$$

Proof: We use Corollary 2 to find $R(x)$. The result follows immediately by computing the digit difference of $R(x)$ in each case. □

Corollary 4: Let x be an ordered 4-digit base g number. If $R^i(x) = 0$ for some $i \geq 1$, then $R^2(x) = 0$.

Proof: Let d' be the digit difference of $R(x)$. Since $R^i(x) = 0$, its digit difference is 0. Applying Corollary 3, repeatedly if necessary, gives $d' = 0$. By Corollary 2, $R^2(x) = 0$. □

Since $R(0) = 0$, there is always one SSR cycle which contains the single number 0. We will call this the **zero cycle** and denote it by $\langle 0 \rangle$. By Corollary 4, we know that if x leads to the zero cycle, it will do so in two or fewer steps. Corollary 3 tells us that there are other numbers which do not lead to the zero cycle. Consequently, the SSR routine always has at least one nonzero cycle.

A RELATED FUNCTION

Suppose x generates a nonzero SSR cycle of length m. Since $R^m(x) = x$, the digit differences of x are equal and nonzero. Moreover, by Corollary 3, if the digit difference of x is $d \neq g/2$, then the digit difference of $R(x)$ is $|g-2d|$. This observation leads us to consider $|g-2d|$.

The function $F(d) = |g-2d|$, $0 \leq d \leq g$, was studied by this author in another context [9]. Since $0 \leq F(d) \leq g$, iteration of F gives rise to one or more cycles. As we will see, the cycles of F are in a one-to-one correspondence with the nonzero SSR cycles so long as the former do not contain 0, $g/2$, or g.

Before continuing, we consider an example. Earlier we applied the SSR routine to the base 15 number $x = 13\ 12\ 10\ 8$. We found $R(x) = 11\ 11\ 3\ 3$, which has a digit difference of 8. Now,

$$\begin{aligned} F(8) &= |15-16| = 1, \\ F^2(8) &= F(1) = |15-2| = 13, \\ F^3(8) &= F(13) = |15-26| = 11, \\ F^4(8) &= F(11) = |15-22| = 7, \\ F^5(8) &= F(7) = |15-14| = 1. \end{aligned}$$

By Corollary 3, since the digit difference of $R(x)$ is 8, the digit difference of $R^2(x)$ is $F(8)$. More generally, for $i \geq 1$, the digit difference of $R^{i+1}(x)$ is $F^i(8)$. This is confirmed by examining the *diff* column in Table 1 and comparing it with the calculations above. Thus, we see that the F-cycle $\langle 1, 13, 11, 7 \rangle$ corresponds to the SSR cycle $\langle R^2(x), R^3(x), R^4(x), R^5(x) \rangle$.

The relevant properties of F are listed below. Proofs may be found in [9]. We will write $F_g(d)$ in place of $F(d)$ whenever the context requires this elaboration.

Theorem 2: Let k be the nonnegative integer such that $2^k \| g$. Then, for $0 < d < g$, we have:

(a) $F(g/2) = 0$, $F(0) = g$, and $F(g) = g$.

(b) d is in an F-cycle if and only if $2^k \| d$.

(c) For $i > 1$, $i \cdot d$ is in an F_{ig}-cycle if and only if d is in an F_g-cycle. More generally, $\langle i \cdot d_1, i \cdot d_2, \ldots, i \cdot d_n \rangle$ is an F_{ig}-cycle if and only if $\langle d_1, d_2, \ldots, d_n \rangle$ is an F_g-cycle.

(d) Let i be the nonnegative integer such that $2^i \| d$. If $i < k-1$, $2^{i+1} \| F(d)$; if $i = k-1$, $2^{k+1} | F(d)$; if $i > k-1$, $2^k \| F(d)$.

(e) For $i > 1$, $F^i(d)$ is congruent to either $2^i d$ or $-2^i d$ modulo g.

(f) $2^k \| F^i(d)$ when $i > k$.

Several conclusions are immediate from Theorem 2. By (a), 0 and $g/2$ are not contained in an F-cycle. Moreover, $\langle g \rangle$ is an F-cycle of length 1. This cycle is called the **trivial** cycle; all other cycles are **proper** cycles. Part (b) tells us that proper cycles exist if and only if g is not a power of 2. By (c), it suffices to determine cycles for odd g. Additionally, we need only consider those d which are relatively prime to g. We will call cycles containing such d **prime**. All other cycles are **composite** since they may be found using (c). Parts (b) and (f) together imply that $F^i(d)$ is in a cycle whenever $i > k$.

Before continuing, we illustrate the previous theorem and definitions. We begin with $g = 5$. By Theorem 2(b), d is in an F_5-cycle if and only if d is odd. Since $F(1) = |5-2| = 3$ and $F(3) = |5-6| = 1$, the only proper F_5-cycle is $\langle 1, 3 \rangle$. The trivial cycle is $\langle 5 \rangle$.

As a second example, we let $g = 20$. By Theorem 2(b), d is in an F_{20}-cycle if and only if $4 \| d$. Since g is even, each proper cycle is composite and may be derived from an F_5-cycle by multiplying by 4. As we have just shown, the only proper F_5-cycle is $\langle 1, 3 \rangle$. Consequently, $\langle 4, 12 \rangle$ is the only proper F_{20}-cycle.

Before considering a third example, we state another result which was proved in [9]. It tells us how to find the lengths of prime cycles.

Theorem 3: Let g be an odd positive integer and let m be the smallest integer such that 2^m is congruent to either 1 or -1 modulo g. Then each prime F-cycle has length m. Moreover, there are $\phi(g)/2m$ such cycles, where $\phi(g)$ is the Euler Phi function.

We now consider $g = 15$. By Theorem 2(b), d is in an F_{15}-cycle if and only if d is odd. Since $\phi(15) = 8$ and $2^4 \equiv 1 \pmod{15}$, there is one prime cycle of length 4. As we have already seen, this cycle is $\langle 1, 13, 11, 7 \rangle$. The composite F_{15}-cycles, found from the proper F_3- and F_5-cycles using Theorem 2(c), are $\langle 5 \rangle$ and $\langle 3, 9 \rangle$.

THE SSR-CYCLES

We now use the previous results to characterize the SSR cycles.

Lemma 1: Suppose d generates a proper F-cycle of length m. Then there exists a nonzero SSR cycle of length m. Moreover, this cycle is generated by an ordered number whose digit difference is d.

Proof: Since $g/2$ is not in an F-cycle, $F^i(d) \neq g/2$ for $0 \leq i$. Let $x = d\ d\ 0\ 0$. Clearly, x has equal digit differences of d. Hence, the digit differences of $R^i(x)$ are $F^i(d), i \geq 1$. By hypothesis, $F^i(d) \neq F^m(d), 0 < i < m$, and $F^m(d) = d$. This means that $R^i(x) \neq R^m(x)$; further, $R^m(x)$ and x have the same digit differences. Hence, by Corollary 1, $R^{m+1}(x) = R(x)$. Thus, $\langle R(x), \ldots, R^m(x) \rangle$ is an SSR cycle of length m; it is generated by $R^m(x)$, whose digit difference is d. □

Lemma 2: Suppose x generates a nonzero SSR cycle of length m. Let d be the digit difference of x. If $F^i(d) \neq g/2, 0 \leq i < m$, then d generates a proper F-cycle of length m.

Proof: The digit difference of $R^i(x)$ is $F^i(d) \neq g/2, i \geq 1$. By hypothesis, $R^m(x) = x$; hence, $F^m(d) = d$ and d generates a proper F-cycle. If $F^i(d) = d$ for some i, $0 < i < m$, then by Corollary 1, $R^{i+1}(x) = R(x)$. But then the SSR cycle would have length $i < m$. Thus, the F-cycle generated by d has length m. □

Theorem 4: Suppose $g = 2^k \cdot t$, where $0 \leq k$ and t is an odd positive integer greater than 1. Let x be an ordered 4-digit base g number with equal digit differences of d. Then x generates an SSR cycle of length m if and only if d generates an F-cycle of length m.

Proof: By Lemmas 1 and 2, we need only consider the case when x is an ordered number with equal digit differences of $g/2$. Suppose that x generates a nonzero SSR cycle of length m. Then, by Corollary 3, the digit difference of $R(x)$ is 1. Thus, for $1 < i \leq m$, the digit differences of $R^i(x)$ is $F^{i-1}(1)$. Since $R^m(x) = x, F^{m-1}(1) = g/2 = 2^{k-1} \cdot t$. By Theorem 2(e), $F^{m-1}(1)$ is congruent to either 2^{m-1} or -2^{m-1} modulo g. As a shorthand notation, we will write $F^{m-1}(1) \equiv \pm 2^{m-1}$ (mod g). Hence, $2^{k-1} \cdot t \equiv \pm 2^{m-1}$ (mod $2^k \cdot t$). Since t is an odd positive integer greater than 1, we have reached a contradiction. □

Theorem 5: Suppose $g = 2^k$, where $0 < k$. Then there is exactly one nonzero SSR cycle. This cycle has length k and contains:

$$
\begin{aligned}
w &= 1\ 1\ 0\ 0 && \text{if } g = 2,\\
w &= 2^k - 2^{k-2}\ \ 2^k - 2^{k-2} - 1\ \ 2^{k-2}\ \ 2^{k-2} - 1 && \text{if } g > 2.
\end{aligned}
$$

Proof: Since g is a power of 2, there are no proper F-cycles. Hence, by Lemma 2, there is at most one nonzero SSR cycle. Moreover, it must be generated by an ordered number whose digit difference is $g/2$.

If $g = 2$, then $\langle w \rangle$ is a nonzero SSR cycle, since $R(w) = w$. If $g > 2$, the digit difference of w is $2^{k-1} = g/2$. By Corollary 2,

$$R(w) = 2^{k-1}\ \ 2^{k-1}\ \ 2^{k-1} - 1\ \ 2^{k-1} - 1$$

and

$$R^{i+2}(w) = 2^k - 2^i\ \ 2^k - 2^i - 1\ \ 2^i\ \ 2^i - 1, \quad 0 \leq i \leq k - 2.$$

Thus, $R^k(w) = w$. □

Theorems 4 and 5 completely characterize the nonzero SSR cycles. If g is not a power of 2, then the nonzero SSR cycles are in a one-to-one correspondence with the proper F-cycles. The cycle lengths can be determined using Theorems 2(c) and 3. The numbers that generate the SSR cycles can be found from the F-cycles using Corollary 2. If g is the k^{th} power of 2, then there is exactly one nonzero SSR cycle; it has length k.

Returning to our base 15 example, previously we found that the F-cycles are $\langle 1,13,11,7\rangle$, $\langle 5\rangle$, and $\langle 3,9\rangle$. The SSR cycle which corresponds to the first F-cycle is given in Table 1. The SSR cycle which corresponds to the F-cycle $\langle 5\rangle$ is $\langle u\rangle$, where the digit difference of u is 5; by Corollary 2, $u = 10\ 9\ 5\ 4$. The SSR cycle which corresponds to the F-cycle $\langle 3,9\rangle$ is $\langle v, R(v)\rangle$, where the digit difference of v is 3; by Corollary 2, $v = 9\ 8\ 6\ 5$.

WHEN IS $R^i(x)$ IN A CYCLE?

We now consider the following question. If x is an ordered 4-digit base g number, what is the smallest i such that $R^i(x)$ is in a cycle?

Lemma 3: Let k be the nonnegative integer such that $2^k \| g$. For all d satisfying $0<d<g$, $F^{k+1}(d)$ is in an F-cycle. Moreover, $F^k(1)$ is not in an F-cycle.

Proof: The first statement follows immediately using parts (b) and (f) of Theorem 2. For the second, we use Theorem 2(d) to show that $2^j \| F^j(1)$ for $j \le k-1$ and $2^{k+1} | F^k(1)$. Hence, by Theorem 2(b), $F^k(1)$ is not in an F-cycle. □

Theorem 6: Suppose $g = 2^k \cdot t$, where $0 \le k$ and t is an odd positive integer greater than 1. Let x be an ordered 4-digit base g number. Then $R^{2k+2}(x)$ is in an SSR cycle. Moreover, there exists an ordered 4-digit base g number y such that $R^{2k+1}(y)$ is not in an SSR cycle.

Proof: Let d be the digit difference of $R(x)$. If $d=0$, then $R^2(x)=0$. Hence, we can assume $d \ne 0$.

First, we consider the case when g is odd. The digit difference of $R^2(x)$ is $F(d) = |g-2d|$. Since $F(d)$ is odd, it is in a proper F-cycle. Hence, by Theorem 4, $R^2(x)$ is in a nonzero SSR cycle. Now consider $y = 1\ 0\ 0\ 0$. Calculating $R(y)$, we find:

$$R(y) = g-1\ \ g-1\ \ 0\ \ 0.$$

The digit difference of $R(y)$ equals $g-1$, which is even. Since $g-1$ is not in an F-cycle, $R(y)$ is not in an SSR cycle, by Theorem 4.

Now suppose that g is even; i.e., $k>0$. If $F^i(d) \ne g/2$ for $0 \le i$, then $F^i(d)$ is the digit difference of $R^{i+1}(x)$. By Lemma 3, $F^{k+1}(d)$ is in a proper F-cycle. Hence, $R^{k+2}(x)$ is in an SSR cycle. On the other hand, suppose that $F^i(d) = g/2$ for some i, $0 \le i \le k-1$. Then the digit difference of $R^{i+2}(x)$ is 1. Consequently, $F^j(1)$ is the digit difference of $R^{i+j+2}(x)$ for $j \le k$. Since $F^k(1)$ is not in a proper F-cycle, $R^{i+k+2}(x)$ is not in an SSR cycle, by Theorem 4. However, again by Theorem 4, because $F^{k+1}(1)$ is in a proper F-cycle, $R^{i+k+3}(x)$ is in an SSR cycle. Consequently, for all x, $R^{2k+2}(x)$ is in an SSR cycle.

We now consider $y = t\ 0\ 0\ 0$. The digit differences of y are t and 0. By Theorem 1, the digit difference of $R(y)$ is $d = 2^k \cdot t - t$. By induction, it is easily established that

$$F^i(d) = 2^k \cdot t - 2^i \cdot t, \quad 0 \le i \le k-1.$$

Hence, $F^{k-1}(d) = g/2$. By the results established above, $R^{2k+1}(y)$ is not in an SSR cycle. □

Lemma 4: Suppose $g = 2^k$, where $1 < k$. For all d satisfying $0 < d < g$, there exists an integer i such that $F^i(d) = g/2$ for $0 \le i \le k-2$.

Proof: Let j be the integer such that $2^j \| d$; of course, $0 \le j \le k-1$. Then $2^{k-1-j} \cdot d \equiv 2^{k-1}$ (mod 2^k). Since $F^{k-1-j}(d) \equiv \pm 2^{k-1-j} \cdot d$ (mod g), $F^{k-1-j}(d) = g/2$.

If $k = 2$, then $F^0(g-1) = g-1 \ne 2$. If $k > 2$, for $0 \le i \le k-2$,

$$F^i(g-1) = F^i(2^k - 1) = 2^k - 2^i \ne 2^{k-1}. \ \square$$

Theorem 7: Suppose $g = 2^k$, where $1 < k$. Let x be an ordered 4-digit base g number. Then $R^k(x)$ is in an SSR cycle. Moreover, there exists an ordered 4-digit base g number y such that $R^{k-1}(y)$ is not in an SSR cycle.

Proof: Let d be the digit difference of $R(x)$. If $d = 0$, then $R^2(x) = 0$. Hence, we can assume $d \ne 0$. By Lemma 4, $F^i(d) = g/2$ for some i, $0 \le i \le k-1$. By Theorem 5, this implies that $R^{i+1}(x)$ is in the nonzero SSR cycle. Hence, for all x, $R^k(x)$ is in an SSR cycle.

We now consider $y = 1\ 0\ 0\ 0$. Calculating $R(y)$, we find

$$R(y) = g-1 \ \ g-1 \ \ 0 \ \ 0.$$

The digit difference of $R(y)$ is $g-1$. By Lemma 4, $F^i(g-1) \ne g/2$ for $i \le k-2$. Hence, $R^{k-1}(y)$ is not in the SSR cycle. $\square$

WHEN DOES THE SSR ROUTINE YIELD A CONSTANT?

Finally, are there bases g for which the SSR routine yields a single, nonzero constant?

Lemma 5: Suppose $1 \le d < g$. Then $\langle d \rangle$ is the only proper F-cycle if and only if $d = g/3$, where $g = 2^k \cdot 3$ for some $0 \le k$.

Proof: First, suppose that $\langle d \rangle$ is the only proper F-cycle. Since $F(d) = d, |g - 2d| = d$ and $d = g/3$. To prove that g has the stated form, we write $g = 2^k \cdot 3 \cdot t$, where t is odd. If $1 < t$, then, by Theorem 2(b), $2^k \cdot 3$ is in a proper F-cycle. But this implies there is a second proper F-cycle. Consequently, $t = 1$ and $g = 2^k \cdot 3$. The converse is easily established using Theorem 2(b). $\square$

Theorem 8: Suppose $g = 2^k \cdot 3$, where $0 \le k$. Let

$$z = 2^{k+1} \ \ 2^{k+1} - 1 \ \ 2^k \ \ 2^k - 1.$$

If x is an ordered 4-digit base g number such that $R^2(x) \ne 0$, then $R^{2k+2}(x) = z$.

Proof: By Lemma 5, there is only one proper F-cycle, $\langle 2^k \rangle$. Consequently, by Theorem 4, there is only one nonzero SSR cycle and it has length 1. This cycle is $\langle z \rangle$, since $R(z) = z$. By Theorem 6, $R^{2k+2}(x)$ is in a cycle; hence, $R^{2k+2}(x) = z$. $\square$

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