ON SOME PROPERTIES OF GENERALIZED HERMITE POLYNOMIALS

Gospava Djordjević
University of Niš, Faculty of Technology, 16000 Leskovac, Yugoslavia
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1. INTRODUCTION

The Hermite polynomials belong to the system of classical orthogonal polynomials (see [3], [6]). The following properties of these polynomials are well known: the orthogonal property, differential equation, Rodrigues representation, three-term recurrence relation. In 1990, P. R. Subramanian [5] studied a class of Hermite polynomials $H_n(x)$ in the sense that one of the above-mentioned four properties implies the other three.

In [4], H. M. Srivastava defined a class of generalized Hermite polynomials $\{\gamma_n^m(x)\}_{n=0}^{\infty}$ by the generating function

$$e^{mxt-t^m}=\sum_{n=0}^{\infty}\gamma_n^m(x)t^n.$$

2. THE POLYNOMIALS $h_{n,m}(x)$

In this paper, we consider the polynomials $\{h_{n,m}(x)\}_{n=0}^{\infty}$ defined by $h_{n,m}(x)=\gamma_n^m(2x/m)$. Their generating function is given by

$$F(x,t)=e^{2xt-t^m}=\sum_{n=0}^{\infty}h_{n,m}(x)t^n. \tag{2.1}$$

Note that $h_{n,2}(x)=H_n(x)/n!$ (Hermite polynomials).

Expanding the left-hand side of (2.1), we obtain the following explicit formula:

$$h_{n,m}(x)=\sum_{k=0}^{[n/m]}(-1)^k\frac{(2x)^{n-mk}}{k!(n-mk)!}. \tag{2.2}$$

By differentiating (2.1) with respect to t and comparing the corresponding coefficients, we obtain the following three-term relation:

$$nh_{n,m}(x)=2xh_{n-1,m}(x)-mh_{n-m,m}(x),\quad n\ge m\ge 1. \tag{2.3}$$

The starting polynomials are

$$h_{n,m}(x)=\frac{(2x)^n}{n!},\quad n=0,1,\dots,m-1. \tag{2.4}$$

By differentiating (2.2) with respect to x, one by one, s times, we get

$$D^s h_{n,m}(x)=2^s h_{n-s,m}(x),\quad n\ge s\ge 1,\ \ D^s\equiv d^s/dx^s. \tag{2.5}$$

For $s=1$, (2.5) is

$$Dh_{n,m}(x)=2h_{n-1,m}(x),\quad n\ge 1. \tag{2.6}$$

For $s = m-1$, (2.5) becomes

$$D^{m-1}h_{n,m}(x) = 2^{m-1}h_{n+1-m,m}(x), \quad n \geq m-1. \tag{2.7}$$

Now, from (2.3) and (2.7), we obtain

$$nh_{n,m}(x) = \left[2x - \frac{m}{2^{m-1}}D^{m-1}\right]h_{n-1,m}(x), \quad n \geq 1, \tag{2.8}$$

where D is the differential operator d/dx.

If $m = 2$, the relation (2.8) becomes (see [1]) $H_n(x) = (2x - D)H_{n-1}(x)$, $n \geq 1$.
A very interesting relation now follows:

$$h_{n,m}(x) = \frac{f^{-1}}{n!}\left[2x - \frac{m}{2^{m-1}}D^{m-1} + \frac{m}{2^{m-1}}\sum_{k=0}^{m-2}(m-k)_k D^{m-1-k}(f)\sum_{j=0}^{k}\frac{D^{k-j}(f^{-1})}{j!(k-j)!}D^j\right]^n f, \tag{2.9}$$

where $f(x)$ is any differentiable function not identically zero, $D^s \equiv d^s/dx^s$, and $(\lambda)_n = \lambda(\lambda+1)$ ... $(\lambda+n-1)$ is the Pochhammer symbol (see [2], [3]).

3. EQUIVALENCE OF (2.9) AND OTHER RELATIONS

First, we shall prove the relation (2.9). Let $f(x)$ be any differentiable function not identically zero. From (2.8), we find:

$$\begin{aligned} fh_{n,m}(x) &= \frac{f}{n}\left[2x - \frac{m}{2^{m-1}}D^{m-1}\right]h_{n-1,m}(x) \\ &= \frac{1}{n}\left[2x - \frac{m}{2^{m-1}}D^{m-1} + \frac{m}{2^{m-1}}\sum_{k=0}^{m-2}(m-k)_k D^{m-1-k}(f)\sum_{j=0}^{k}\frac{D^{k-j}(f^{-1})}{j!(k-j)!}D^j\right]\{fh_{n-1,m}(x)\}. \end{aligned} \tag{3.1}$$

Iteration of (3.1) yields

$$fh_{n,m}(x) = \frac{1}{n!}\left[2x - \frac{m}{2^{m-1}}D^{m-1} + \frac{m}{2^{m-1}}\sum_{k=0}^{m-2}(m-k)_k D^{m-1-k}(f)\sum_{j=0}^{k}\frac{D^{k-j}(f^{-1})}{j!(k-j)!}D^j\right]^n f, \quad n \geq 1, \tag{3.2}$$

since $h_{0,m}(x) = 1$. However, (3.2) is also true for $n = 0$. The relation (2.9) follows immediately.

From (2.9) and $f(x) = 1$, we get the following beautiful relation:

$$h_{n,m}(x) = \frac{1}{n!}\left[2x - \frac{m}{2^{m-1}}D^{m-1}\right]^n 1, \quad n \geq 0. \tag{3.3}$$

If $m = 2$, (3.3) becomes (see [1]) $H_n(x) = [2x - D]^n 1$, $n \geq 0$. If $m = 3$, then (2.9) becomes

$$h_{n,3}(x) = \frac{f^{-1}}{n!}\left[2x - \frac{3}{4}D^2 + \frac{3}{4}f^{-1}\{D^2 f\} + \frac{3}{2}\{Df\}\{Df^{-1}\} + \frac{3}{2}f^{-1}\{Df\}D\right]^n f, \quad n \geq 0. \tag{3.4}$$

If $f(x) = e^{-x^3}$, relation (3.4) yields

$$h_{n,3}(x) = \left(-\frac{1}{4}\right)^n \frac{e^{x^3}}{n!}[3D^2 + 10x + 27x^4 + 18x^2 D]^n e^{-x^3}, \quad n \geq 0.$$

Now, we shall show that (2.9) is a spring for developing the properties of $h_{n,m}(x)$. First, we prove (2.8), starting from (2.9):

$$h_{n,m}(x) = \frac{f^{-1}}{n!}\left[2x - \frac{m}{2^{m-1}}D^{m-1} + \frac{m}{2^{m-1}}\sum_{k=0}^{m-2}(m-k)_k D^{m-1-k}(f)\sum_{j=0}^{k}\frac{D^{k-j}(f^{-1})}{j!(k-j)!}D^j\right]^n f$$

$$= \frac{1}{n!}\left[2x - \frac{m}{2^{m-1}}D^{m-1}\right]^n 1 = \frac{1}{n}\left[2x - \frac{m}{2^{m-1}}D^{m-1}\right]h_{n-1,m}(x), \quad n \geq 1.$$

Hence, we get (2.8).

From (2.3) with $n+1$ substituted for n, and using (2.5), (2.6), and (2.8), we find

$$(n+1)Dh_{n+1,m}(x) = D\left[2x - \frac{m}{2^{m-1}}D^{m-1}\right]h_{n,m}(x)$$

$$= 2h_{n,m}(x) + \left[2x - \frac{m}{2^{m-1}}D^{m-1}\right]Dh_{n,m}(x) = 2(n+1)h_{n,m}(x).$$

Thus, we obtain the following differential recurrence relation:

$$Dh_{n+1,m}(x) = 2h_{n,m}(x). \tag{3.5}$$

Now, we shall prove the three-term recurrence relation (2.3). From (2.7) and (2.8), we get

$$nh_{n,m}(x) = 2xh_{n-1,m}(x) - \frac{m}{2^{m-1}}D^{m-1}h_{n-1,m}(x)$$

$$= 2xh_{n-1,m}(x) - mh_{n-m,m}(x), \quad n \geq m \geq 1.$$

The relation (2.3) follows from the last equality.

By differentiating (2.8) with $n+1$ substituted for n, and using (3.5), we obtain

$$\begin{aligned}(n+1)Dh_{n+1,m}(x) &= D\left[2x - \frac{m}{2^{m-1}}D^{m-1}\right]h_{n,m}(x)\\ &= 2h_{n,m}(x) + \left[2x - \frac{m}{2^{m-1}}D^{m-1}\right]Dh_{n,m}(x).\end{aligned} \tag{3.6}$$

Next, from (3.6), we get the following differential equation:

$$\left[\frac{m}{2^{m-1}}D^m - 2xD + 2n\right]h_{n,m}(x) = 0. \tag{3.7}$$

For $m = 2$ or 3, equation (3.7) becomes

$$[D^2 - 2xD + 2n]H_n(x) = 0, \quad n \geq 0,$$

and

$$\left[\frac{3}{4}D^3 - 2xD + 2n\right]h_{n,3}(x) = 0, \quad n \geq 0.$$

Note that the first equation above is the Hermite differential equation.

Now, we show that (2.1) can be derived from the recurrence relation as follows (see [5]). Assume the existence of a generating function of the form

$$F(x, t) = \sum_{n=0}^{\infty} h_{n,m}(x)t^n. \tag{3.8}$$

Differentiate $F(x,t)$ with respect to t and, using (2.3) and (2.4), develop the following first-order differential equation for $F(x,t)$:

$$F^{-1}(\partial F / \partial t) = 2x - mt^{m-1}. \tag{3.9}$$

Now, we integrate both sides of (3.9) with respect to t, from 0 to t, to obtain

$$F(x, t) = F(x, 0)e^{2xt-t^m}. \tag{3.10}$$

Since $F(x, 0) = h_{0,m}(x) = 1$, by (2.4), it follows that $F(x,t) = e^{2xt-t^m}$.

Finally, we shall prove in this section that the polynomial $h_{n,m}(x)$ is a solution of the differential equation (3.7).

Assume that the polynomial $y = \Sigma_{k=0}^{n} a_k \cdot x^{n-k}$ is a solution of equation (3.7). Then,

$$Dy = \sum_{k=0}^{n-1}(n-k)a_k \cdot x^{n-1-k}, \tag{3.11}$$

and

$$D^m y = \sum_{k=0}^{n-m}(n+1-m-k)_m \cdot a_k \cdot x^{n-m-k}. \tag{3.12}$$

If we substitute (3.11) and (3.12) into equation (3.7), we get

$$\frac{m}{2^{m-1}}\sum_{k=m}^{n}(n+1-k)_m \cdot a_{k-m} \cdot x^{n-k} - 2\sum_{k=0}^{n}(n-k)a_k \cdot x^{n-k} + 2n\sum_{k=0}^{n} a_k \cdot x^{n-k} = 0. \tag{3.13}$$

From (3.13), we obtain

$$\sum_{k=m}^{n}\left[\frac{m}{2^{m-1}}(n+1-k)_m \cdot a_{k-m} + 2ka_k\right]x^{n-k} + \sum_{k=0}^{m-1} 2ka_k \cdot x^{n-k} = 0. \tag{3.14}$$

Next, from (3.14), we find

$$ka_k = 0, \quad k = 0, 1, 2, \ldots., m-1, \tag{3.15}$$

and

$$a_k = -\frac{m(n+1-k)_m}{2^m k}a_{k-m}, \quad k \geq m. \tag{3.16}$$

Finally, from (3.15), (3.16), and $a_0 = 2^n / n!$, using induction, we can show that the polynomial $y = \Sigma_{k=0}^{n} a_k \cdot x^{n-k}$ has the following form:

$$y = \sum_{k=0}^{[n/m]}(-1)^k \frac{(2x)^{n-mk}}{k!(n-mk)!}. \tag{3.17}$$

Comparing (3.17) with (2.2), we see that the polynomial y is the generalized Hermite polynomial $h_{n,m}(x)$.

4. RELATION $h_{n,m}(x) = (2^n / n!)\{\exp[-D^m / 2^m]\}x^n$

In this section, we prove the following relation

$$h_{n,m}(x) = \frac{2^n}{n!}\{\exp[-D^m / 2^m]\}\, x^n. \tag{4.1}$$

Note that the operator $\exp[-D^m / 2^m]$ has the following expansion:

$$\exp[-D^m / 2^m] = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!} \frac{D^{ms}}{2^{ms}}. \tag{4.2}$$

Since

$$D^{ms}x^n = \begin{cases} [n!/(n-ms)!]x^{n-ms}, & n \geq ms \quad (s \leq [n/m]), \\ 0, & n < ms. \end{cases} \tag{4.3}$$

The relation (4.1) follows from (2.2), using (4.2) and (4.3).

For $m = 2$, (4.1) has the form (see [2])

$$H_n(x) = 2^n\{\exp[-D^2 / 4]\}x^n;$$

for $m = 3$, (4.1) becomes

$$h_{n,3}(x) = \frac{2^n}{n!}\{\exp[-D^3 / 8]\}x^n.$$

Remark: We can classify the starting points into two distinct groups (see [5]): (a) full self-contained springs and (b) associated springs. The generating function (2.1) and the relations (2.2), (2.9), (3.3), and (4.1) belong to category (a). These springs completely specify the generalized Hermite polynomials $h_{n,m}(x)$. The differential equation (3.7), the recurrence relation (2.3), the differential recurrence relation (3.5), and the relation (2.8) belong to category (b) because they require supplementary conditions to specify the generalized Hermite polynomials $h_{n,m}(x)$ fully.

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