MINMAX POLYNOMIALS

A. F. Horadam
The University of New England, Armidale, Australia 2351
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1. INTRODUCTION

Background

MinMax numbers $\{M_n\}$, and their subsidiary numbers $\{N_n\}$, for Pell numbers $\{P_n\}$ were studied in some detail in [2]. They are those positive integers for which the minimal and maximal representations by Pell numbers coincide.

Analogous results for the MinMax numbers $\{\mathcal{Q}_n\}$, and their subsidiary numbers $\{\mathfrak{R}_n\}$, for the modified Pell numbers $\{q_n\}$ have been obtained in [3]. ($q_n = \frac{1}{2}Q_n$, where Q_n are the Pell-Lucas numbers [4].)

Our motivation in this paper is to extend these MinMax number systems to their algebraic polynomial counterparts $\{M_n(x)\}, \{N_n(x)\}, \{\mathcal{Q}_n(x)\}$, and $\{\mathfrak{R}_n(x)\}$, and to analyze their properties. When $x = 1$, the MinMax numbers $\{M_n(1)\} = \{M_n\}$, etc., are naturally specified.

Pell polynomials $\{P_n(x)\}, n \geq 0$, are defined recursively [4] by

$$P_{n+2}(x) = 2xP_{n+1}(x) + P_n(x), \quad P_0(x) = 0, \ P_1(x) = 1, \tag{1.1}$$

while the modified Pell polynomials $\{q_n(x)\}, n \geq 0$, are similarly defined by

$$q_{n+2}(x) = 2xq_{n+1}(x) + q_n(x), \quad q_0(x) = 1, \ q_1(x) = x. \tag{1.2}$$

A useful connective between (1.1) and (1.2) is $q_n(x) = xP_n(x) + P_{n-1}(x)$.

Detailed information on the properties of, and interrelations between, $\{P_n(x)\}$ and $\{Q_n(x) = 2q_n(x)\}$ appear in [4] and [5], including lists of some of these polynomials. To conserve space, we assume that these data are accessible to the reader.

Just as there is the connection [2] between M_n and P_n, so there is the polynomial nexus

$$M_n(x) = \sum_{i=1}^{n} P_i(x). \tag{1.3}$$

The Sequence $\{q_n^*(x)\}$

Allied to $\{q_n(x)\}$ is the polynomial sequence $\{q_n^*(x)\}$ defined by the recurrence relation

$$q_{n+2}^*(x) = 2xq_{n+1}^* + q_n^*(x), \quad q_0^*(x) = 1, \ q_1^*(x) = 1. \tag{1.4}$$

Whereas $q_1(x) = x$, here $q_1^*(x) = 1$. Consult Table 1.

Putting $x = 1$ in $q_n^*(x)$, we find that $q_n^* = q_n$. Expressed otherwise, both $\{q_n^*(x)\}$ and $\{q_n(x)\}$ are polynomial generalizations of the modified Pell numbers $\{q_n\}$.

By standard methods, we derive the generating function

$$[1-(1-2x)y][1-(2xy+y^2)]^{-1} = \sum_{i=0}^{\infty} q_i^* y^i \tag{1.5}$$

and the Binet form

$$q_n^*(x) = \frac{(1-\beta)\alpha^n - (1-\alpha)\beta^n}{\alpha - \beta} \tag{1.6}$$

where [2]

$$\begin{cases} \alpha = x + \Delta \\ \beta = x - \Delta \end{cases} \quad \text{where } \Delta = \sqrt{x^2+1} \tag{1.7}$$

leading to

$$q_n^*(x) = P_n(x) + P_{n-1}(x). \tag{1.8}$$

For convenience, in (1.7) we employ the abbreviated symbolism $\alpha \equiv \alpha(x), \beta \equiv \beta(x), \Delta \equiv \Delta(x)$.

Using (1.7) and (1.8) in conjunction with the Binet form and Simson formula for $\{P_n(x)\}$, we have eventually the Simson analogue for $\{q_n^*(x)\}$:

$$q_{n+1}^* q_{n-1}^* - (q_n^*(x))^2 = (-1)^{n-1} 2x. \tag{1.9}$$

TABLE 1

$q_0^*(x) = 1$	$q_0(x) = 1$
$q_1^*(x) = 1$	$q_1(x) = x$
$q_2^*(x) = 2x+1$	$q_2(x) = 2x^2+1$
$q_3^*(x) = 4x^2+2x+1$	$q_3(x) = 4x^3+3x$
$q_4^*(x) = 8x^3+4x^2+4x+1$	$q_4(x) = 8x^4+8x^2+1$
$q_5^*(x) = 16x^4+8x^3+12x^2+4x+1$	$q_5(x) = 16x^5+20x^3+5x$
$q_6^*(x) = 32x^5+16x^4+32x^3+12x^2+6x+1$	....
$q_7^*(x) = 64x^6+32x^5+80x^4+32x^3+24x^2+6x+1$	
$q_8^*(x) = 128x^7+64x^6+192x^5+80x^4+80x^3+24x^2+8x+1$	
....	

2. MINMAX POLYNOMIALS $\{M_n(x)\}$

Define the polynomials $\{M_n(x)\}$, $n \geq 0$, by the recurrence relation

$$M_{n+2}(x) = 2xM_{n+1}(x) + M_n(x) + 1, \quad M_0(x) = 0, \; M_1(x) = 1. \tag{2.1}$$

Polynomials $\{M_n(x)\}$ may be called the *MinMax polynomials* for the Pell numbers.

Letting $x = 1$ gives us the MinMax numbers $\{M_n\}$ for the Pell numbers [2].

Table 2 displays the first few polynomials of $\{M_n(x)\}$.

That (1.3) and (2.1) are in conformity may be deduced by exploiting the defining recurrence relation (1.1) for $\{P_n(x)\}$ and recalling (1.1) that $P_1(x) = 1$.

It is a straightforward procedure by a standard technique to obtain the generating function for $\{M_n(x)\}$:

$$[1-(2x+1)y+(2x-1)y^2+y^3]^{-1} = \sum_{i=1}^{\infty} M_i(x) y^{i-1}. \tag{2.2}$$

From (1.3) and [4, (2.15)], we may express $M_n(x)$ explicitly by means of the double summation

$$M_n(x)=\sum_{i=1}^{n}\left(\sum_{k=0}^{\left[\frac{n-1}{2}\right]}\binom{i-k-1}{k}(2x)^{i-2k-1}\right). \tag{2.3}$$

Illustration of (2.3):

$$M_{11}(x)=1024x^{10}+512x^9+2560x^8+1152x^7+2304x^6 \\ +896x^5+896x^4+280x^3+140x^2+30x+6$$

(on calculation), which is readily verifiable from Table 2 and (2.1).

TABLE 2. The MinMax Polynomials $M_n(x)$ ($n=0,1,2,\ldots,10$)

$M_0(x)=0$
$M_1(x)=1$
$M_2(x)=2x+1$
$M_3(x)=4x^2+2x+2$
$M_4(x)=8x^3+4x^2+6x+2$
$M_5(x)=16x^4+8x^3+16x^2+6x+3$
$M_6(x)=32x^5+16x^4+40x^3+16x^2+12x+3$
$M_7(x)=64x^6+32x^5+96x^4+40x^3+40x^2+12x+4$
$M_8(x)=128x^7+64x^6+224x^5+96x^4+120x^3+40x^2+20x+4$
$M_9(x)=256x^8+128x^7+512x^6+224x^5+336x^4+120x^3+80x^2+20x+5$
$M_{10}(x)=512x^9+256x^8+1152x^7+512x^6+896x^5+336x^4+280x^3+80x^2+30x+5$
....

Combining (1.3) and [4, (2.11)], we derive

$$M_n(x)=\frac{P_{n+1}(x)+P_n(x)-1}{2x}=\frac{q^*_{n+1}(x)-1}{2x} \tag{2.4}$$

yielding the Binet form

$$M_n(x)=\frac{\alpha^n(1+\alpha)-\beta^n(1+\beta)-\Delta}{2x\Delta}. \tag{2.5}$$

A characteristic feature of $\{M_n(x)\}$ is the Simson formula

$$M_{n+1}(x)M_{n-1}(x)-M_n^2(x)=\frac{P_n(x)-P_{n+1}(x)+(-1)^n}{2x}. \tag{2.6}$$

Other derivations of interest in the MinMax theory include

$$M_n(x)-M_{n-1}(x)=P_n(x), \tag{2.7}$$

$$M_n(x)-M_{n-2}(x)=q^*_n(x) \quad \text{by (2.7), (1.8),} \tag{2.8}$$

$$M_n(x)+M_{n-1}(x)=\frac{P_{n+1}(x)+(1-x)P_n(x)-1}{x} \quad \text{by (2.4), (1.1)} \\ (=\mathcal{Q}(x) \quad \text{by (4.3)),} \tag{2.9}$$

$$M_n(x)+M_{n-2}(x)=\frac{(1+x)P_n(x)+(1-x)P_{n-1}-1}{x} \quad \text{by (2.7), (2.9)} \qquad (2.10)$$
$$(=N_{n-1} \quad \text{by (3.1)}).$$

3. THE SUBSIDIARY MINMAX POLYNOMIALS $\{N_n(x)\}$

Next, we introduce a sequence of polynomials $\{N_n(x)\}$ associated with $\{M_n(x)\}$ which we define thus ($n \geq 1$):

$$N_n(x)=M_{n+1}(x)+M_{n-1}(x), \quad N_0(x)=1. \qquad (3.1)$$

These polynomials $\{N_n(x)\}$ may be called the *subsidiary polynomials* of $\{M_n(x)\}$ for the Pell numbers. Table 3 lists the first few of them. See also (2.10).

TABLE 3. The Subsidiary MinMax Polynomials $N_n(x)$ ($n = 0, 1, 2, \ldots, 9$)

$N_0(x)=1$
$N_1(x)=2x+1$
$N_2(x)=4x^2+2x+3$
$N_3(x)=8x^3+4x^2+8x+3$
$N_4(x)=16x^4+8x^3+20x^2+8x+5$
$N_5(x)=32x^5+16x^4+48x^3+20x^2+18x+5$
$N_6(x)=64x^6+32x^5+112x^4+48x^3+56x^2+18x+7$
$N_7(x)=128x^7+64x^6+256x^5+112x^4+160x^3+56x^2+32x+7$
$N_8(x)=256x^8+128x^7+576x^6+256x^5+432x^4+160x^3+120x^2+32x+9$
$N_9(x)=512x^9+256x^8+1280x^7+576x^6+1120x^5+432x^4+400x^3+120x^2+50x+9$
....

When $x=1$, the numerical specializations are the *subsidiary numbers* $\{N_n\}$ investigated in [2].

For the criterion $N_0(x)=1$ to prevail in (3.1), we necessarily have $M_{-1}(x)=0$, obtainable by extension of (2.1) to the value of $n=-1$.

Immediately from (3.1) with (2.1) flows the consequence

$$N_{n+2}(x)=2xN_{n+1}(x)+N_n(x)+2 \quad (n \geq 0), \qquad (3.2)$$

which is the recurrence relation for $\{N_n(x)\}$.

The generating function for $\{N_n(x)\}$ is, from (2.2) and (3.1),

$$(1+y^2)[1-(2x+1)y+(2x-1)y^2+y^3]^{-1}=\sum_{i=1}^{\infty} N_i(x)y^{i-1}. \qquad (3.3)$$

Explicitly, from (2.3) and (3.1),

$$N_n(x)=2\sum_{i=1}^{n-1}\underbrace{\left(\sum_{k=0}^{[\frac{n-1}{2}]}\binom{i-k-1}{k}(2x)^{i-2k-1}\right)}_{A}+\sum_{i=n}^{n+1}\left[A+\binom{i-[\frac{n+1}{2}]-1}{[\frac{n+1}{2}]}(2x)^{i-2[\frac{n+1}{2}]-1}\right], \qquad (3.4)$$

in which A stands for the second summation in the double summation, as represented symbolically.

Perseverance in calculation with (3.5) leads us to, for instance,

$$\begin{aligned} N_{10}(x) = 1024x^{10} + 512x^9 + 2816x^8 + 1280x^7 + 2816x^6 + 1120x^5 \\ + 1232x^4 + 400x^3 + 220x^2 + 50x + 11, \end{aligned}$$

which may be readily checked from Table 3 and (3.2), or directly from (3.1) in conjunction with Table 2. Recall the expression for $M_{11}(x)$ in the illustration of (2.3).

Equations (2.4) and (3.1) produce

$$\begin{cases} N_n(x) = \dfrac{(1+x)P_{n+1}(x) + (1-x)P_n(x) - 1}{x} \\ \qquad = \dfrac{Q_{n+1}(x) + Q_n(x) - 2}{2x} \end{cases} \tag{3.5}$$

(where $Q_n(x) = P_{n+1}(x) + P_{n-1}(x)$ [4, (2.1)]), leading to the Binet form, see (1.7),

$$N_n(x) = \frac{\alpha^n(1+\alpha) + \beta^n(1+\beta) - 2}{2x}. \tag{3.6}$$

Suitable algebraic manipulation, involving (3.5) and [4], reveals in due course the Simson formula for $\{N_n(x)\}$,

$$N_{n+1}(x)N_{n-1}(x) - N_n^2(x) = \frac{Q_n(x) - Q_{n+1}(x) + 2\Delta^2(-1)^{n+1}}{x}. \tag{3.7}$$

Furthermore, (3.5) with (1.2), in which $Q_n(x) = 2q_n(x)$, reveals that

$$N_n(x) - N_{n-1}(x) = Q_n(x), \tag{3.8}$$

whence

$$N_n(x) - N_{n-2}(x) = Q_{n+1}(x) + Q_n(x). \tag{3.9}$$

Moreover,

$$N_n(x) + N_{n-1}(x) = \frac{Q_{n+1}(x) + (1-x)Q_n(x) - 2}{x} \quad \text{by (3.5)} \tag{3.10}$$

and

$$N_n(x) + N_{n-2}(x) = \frac{(1+x)Q_n(x) + (1-x)Q_{n-1}(x) - 2}{x} \quad \text{by (3.8), (3.10).} \tag{3.11}$$

Perusing the polynomial properties in Sections 2 and 3, one is struck by the harmonious balance of those results for $\{M_n(x)\}$ relating to $\{P_n(x)\}$ and similar ones for $\{N_n(x)\}$ relating to $\{Q_n(x)\}$ ("sweet harmony of contrasts"), e.g., compare (2.10) and (3.11).

This mathematical symbiosis does not really transfer to the polynomials to be discussed in Sections 4 and 5, though.

Notice, however, the same direct nexus between results for $\{M_n(x)\}$ in relation to $P_n(x)$ and those for $\{\mathfrak{Q}_n(x)\}$ in relation to $\{q_n^*\}$, e.g., contrast (2.10) and (4.10).

Comparison, e.g., of (3.11) and (4.10) shows the balance between results for $\{N_n(x)\}$ in relation to $\{Q_n(x)\}$ and those for $\{\mathfrak{Q}_n(x)\}$ in relation to $\{q_n^*(x)\}$, thus completing the third "side" of a "triangle" of relationships, i.e., (2.10), (3.11), and (4.10).

Second-order expressions (excepting the Simson analogues) are generally less manageable. Difference of squares such as $M_n^2(x) - M_{n-1}^2(x) = (M_n(x) + M_{n-1}(x))(M_n(x) - M_{n-1}(x))$, etc., are readily derivable from (2.7) and (2.9), but direct calculations and simplifications would otherwise be onerous. Coming to the sum of squares, we discover that

$$M_n^2(x) + M_{n-1}^2(x) = \frac{1}{2x^2}\left[\left\{\frac{1}{2}Q_n(x) + P_{2n}(x) - Q_n(x) - 2P_n(x)\right\} + 1\right] \tag{2.11}$$

and

$$N_n^2(x) - N_{n-1}^2(x) = \frac{1}{2x^2}\left[\left\{\frac{1}{2}Q_{(2n)}(x) + P_{(2n)}(x) - Q_{(n)}(x) - 2P_{(n)}(x)\right\} + 4\right] - 2P_n(x) \tag{3.12}$$

where, in the latter equation, we have used the symbolism, e.g.,

$$P_{(n)}(x) = P_{n+1}(x) + 2P_n(x) + P_{n-1}(x). \tag{3.13}$$

4. THE MINMAX POLYNOMIALS $\{\mathscr{Q}_n(x)\}$

Instead of the MinMax polynomials for the Pell numbers, we now consider the analogous polynomials for the modified Pell numbers.

Define the *MinMax polynomials* $\{\mathscr{Q}_n(x)\}, n \geq 0$, for the modified Pell numbers $\{q_n\}$—see (1.2)—by the recurrence relation

$$\mathscr{Q}_{n+2}(x) = 2x\mathscr{Q}_{n+1}(x) + \mathscr{Q}_n(x) + 2, \quad \mathscr{Q}_0(x) = 0,\ \mathscr{Q}_1(x) = 1. \tag{4.1}$$

Table 4 records the simplest of these polynomials.

TABLE 4. The MinMax Polynomials $\mathscr{Q}_n(x)$ ($n = 0, 1, 2, \ldots, 7$)

$\mathscr{Q}_0(x) = 0$
$\mathscr{Q}_1(x) = 1$
$\mathscr{Q}_2(x) = 2x + 2$
$\mathscr{Q}_3(x) = 4x^2 + 4x + 3$
$\mathscr{Q}_4(x) = 8x^3 + 8x^2 + 8x + 4$
$\mathscr{Q}_5(x) = 16x^4 + 16x^3 + 20x^2 + 12x + 5$
$\mathscr{Q}_6(x) = 32x^5 + 32x^4 + 48x^3 + 32x^2 + 18x + 6$
$\mathscr{Q}_7(x) = 64x^6 + 64x^5 + 112x^4 + 80x^3 + 56x^2 + 24x + 7$
....

Letting $x = 1$, we obtain the MinMax numbers $\{\mathscr{Q}_n\}$ for $\{q_n\}$ given in [3], namely, $\{\mathscr{Q}_n(x)\} = \{1, 4, 11, 28, 69, 168, 407, \ldots\}$.

Without difficulty, we establish the generating function

$$(1+y)[1-(2x+1)y+(2x-1)y^2+y^3]^{-1} = \sum_{i=1}^{\infty}\mathscr{Q}_i(x)y^{i-1}. \tag{4.2}$$

Immediately, we have from (4.2) with (2.2) that [cf. (2.9)]

$$\mathfrak{Q}_n(x) = M_n(x) + M_{n-1}(x). \tag{4.3}$$

With (2.5) substituted in (4.3), there results the Binet form for $\mathfrak{Q}_n(x)$, see (1.7),

$$\mathfrak{Q}_n(x) = \frac{\alpha^{n-1}(1+\alpha)^2 - \beta^{n-1}(1+\beta)^2 - 2\Delta}{2x\Delta} \tag{4.4}$$

which gives, with (1.6),

$$\mathfrak{Q}_n(x) = \frac{q_{n+1}^* + q_n^*(x) - 2}{2x}. \tag{4.5}$$

Compare this with the form for $N_n(x)$ in (3.5).

Using (4.5) along with (1.4) and (1.8), we discover the Simson formula

$$\mathfrak{Q}_{n+1}(x)\mathfrak{Q}_{n-1}(x) - \mathfrak{Q}_n^2(x) = (-1)^{n-1} - \frac{(q_{n+1}^*(x) - q_n^*(x))}{x}. \tag{4.6}$$

It readily follows from (4.5) and (1.4) that

$$\mathfrak{Q}_n(x) - \mathfrak{Q}_{n-1}(x) = q_n^*(x), \tag{4.7}$$

whence

$$\mathfrak{Q}_n(x) - \mathfrak{Q}_{n-2}(x) = q_{n+1}^*(x) + q_n^*(x). \tag{4.8}$$

Also,

$$\mathfrak{Q}_n(x) + \mathfrak{Q}_{n-1}(x) = \frac{1}{x}\{q_{n+1}^*(x) + (1-x)q^*(x) - 2\} \quad \text{by (4.5), (1.4),} \tag{4.9}$$

giving

$$\begin{aligned} \mathfrak{Q}_n(x) + \mathfrak{Q}_{n-2}(x) &= \frac{1}{x}\{(1+x)q_n^*(x) + (1-x)q_{n-1}^* - 2\} \quad &&\text{by (4.7)} \\ &(= \mathfrak{R}_{n-1}(x) &&\text{by (5.1)).} \end{aligned} \tag{4.10}$$

An important result is

$$\mathfrak{Q}_n(x) = \sum_{i=1}^{n} q_i^*(x). \tag{4.11}$$

Proof of (4.11)

$$\begin{aligned} \sum_{i=1}^{n} q_i^*(x) &= \frac{1}{\Delta}\left\{(1-\beta)\sum_{i=1}^{n}\alpha^i - (1-\alpha)\sum_{i=1}^{n}\beta^i\right\} \quad \text{by (1.6)} \\ &= \frac{1}{\Delta}\left\{(1-\beta)\alpha\cdot\frac{1-\alpha^n}{1-\alpha} - (1-\alpha)\beta\cdot\frac{1-\beta^n}{1-\beta}\right\} \\ &= \frac{1}{\Delta}\left\{(1+\alpha)\left(\frac{1-\alpha^n}{1-\alpha}\right) - (1+\beta)\left(\frac{1-\beta^n}{1-\beta}\right)\right\} \\ &= \frac{(1+\alpha)(1-\beta)(1-\alpha^n) - (1-\alpha)(1+\beta)(1-\beta^n)}{\Delta(1-\alpha)(1-\beta)} \end{aligned}$$

$$= \frac{(2+\Delta)(1-\alpha^n)-(2-\Delta)(1-\beta^n)}{-2x\Delta} \quad \text{by (1.7)}$$
$$= \frac{2+\Delta-2+\Delta-[\alpha^n(2+\alpha+\alpha^{-1})-\beta^n(2+\beta+\beta^{-1})]}{-2x\Delta}$$
$$= \frac{\alpha^{n-1}(1+\alpha)^2-\beta^{n-1}(1+\beta)^2-2\Delta}{2x\Delta}$$
$$= \mathfrak{Q}_n(x) \quad \text{by (4.4).}$$

So,

$$\mathfrak{Q}_n(x)=\sum_{i=1}^{n} q_i^*(x) \neq \sum_{i=1}^{n} q_i(x).$$

5. THE SUBSIDIARY MINMAX POLYNOMIALS $\{\mathfrak{R}_n(x)\}$

We now introduce a sequence of polynomials $\{\mathfrak{R}_n(x)\}$ which bears the same relationship to $\{\mathfrak{Q}_n(x)\}$ for $\{q_n(x)\}$ as $\{N_n(x)\}$ bears to $\{M_n(x)\}$ for $P_n(x)$.

Define the *subsidiary MinMax polynomials* $\{\mathfrak{R}_n(x)\}$ of $\{\mathfrak{Q}_n(x)\}$ for $\{q_n(x)\}$ by

$$\mathfrak{R}_n(x)=\mathfrak{Q}_{n+1}(x)+\mathfrak{Q}_{n-1}(x), \quad \mathfrak{R}_0(x)=0, \tag{5.1}$$

whence, by (4.1),

$$\mathfrak{R}_{n+2}(x)=2x\mathfrak{R}_{n+1}(x)+\mathfrak{R}_n(x)+4. \tag{5.2}$$

For the definition to apply for $n \geq 0$, we must have $\mathfrak{Q}_{-1}(x)=-1$. Some of the most elementary of these polynomials are displayed in Table 5.

TABLE 5. The Subsidiary MinMax Polynomials $\mathfrak{R}_n(x)$ ($n = 0, 1, 2, \ldots, 5$)

$\mathfrak{R}_0(x)=0$
$\mathfrak{R}_1(x)=2x+2$
$\mathfrak{R}_2(x)=4x^2+4x+4$
$\mathfrak{R}_3(x)=8x^3+8x^2+10x+6$
$\mathfrak{R}_4(x)=16x^4+16x^3+24x^2+16x+8$
$\mathfrak{R}_5(x)=32x^5+32x^4+56x^3+40x^2+26x+10$
....

Putting $x = 1$ gives $\mathfrak{R}_n(1)=4M_n(1)$, i.e., $\mathfrak{R}_n=4M_n$ as in [3].

It is a relatively effortless exercise to determine the generating function

$$2[(1+x)+(1-x)y][1-(2x+1)y+(2x-1)y^2+y^3]^{-1}=\sum_{i=1}^{\infty}\mathfrak{R}_i(x)y^{i-1} \tag{5.3}$$

leading to

$$\mathfrak{R}_n(x)=2[(1+x)M_n(x)+(1-x)M_{n-1}(x)], \tag{5.4}$$

where we have invoked (2.2).

After a little simplification involving (2.1) in (5.4), this is reducible to [cf. (3.10)]

$$\mathfrak{R}_n(x)=N_n(x)+N_{n-1}(x) \tag{5.5}$$

which by (3.6) ensures the Binet form, see (1.7),

$$\mathfrak{R}_n(x) = \frac{\alpha^{n-1}(1+\alpha)^2 + \beta^{n-1}(1+\beta)^2 - 4}{2x}. \tag{5.6}$$

Equations (5.1) and (4.1) together produce

$$\begin{aligned} \mathfrak{R}_n(x) &= \frac{\mathfrak{Q}_{n+2}(x) - \mathfrak{Q}_{n-2}(x) - 4}{2x} \\ &= \frac{Q_{n+1}(x) + 2Q_n(x) + Q_{n-1}(x) - 4}{2x} \quad \text{by (5.5) and (3.5).} \end{aligned} \tag{5.7}$$

Next,

$$\begin{aligned} \mathfrak{R}_n(x) - \mathfrak{R}_{n-1}(x) &= q^*_{n+1}(x) + q^*_{n-1}(x) && \text{by (5.1), (4.7)} \\ &= Q_n(x) + Q_{n-1}(x) && \text{by (1.8), [4, (2.1)]} \\ &= 2(q_n(x) + q_{n-1}(x)) && \text{since } Q_n(x) = 2q_n(x) \\ &= N_n(x) - N_{n-2}(x) && \text{by (5.5),} \end{aligned} \tag{5.8}$$

leading to expressions for $\mathfrak{R}_n(x) - \mathfrak{R}_{n-2}(x)$.

Moreover,

$$\mathfrak{R}_n(x) + \mathfrak{R}_{n-1}(x) = \frac{\alpha^{n-2}(1+\alpha)^3 + \beta^{n-2}(1+\beta)^3 - 8}{2x} \quad \text{by (5.6)} \tag{5.9}$$

whence

$$\mathfrak{R}_n(x) + \mathfrak{R}_{n-1}(x) = \frac{Q_{n-2}(x) + 3Q_{n-1}(x) + 3Q_n(x) + Q_{n+1}(x) - 8}{2x} \tag{5.10}$$

on using the Binet form [4, (3.31)] for $\{Q_n(x)\}$. An expression for $\mathfrak{R}_n(x) + \mathfrak{R}_{n-2}(x)$ follows by joining (5.8) to (5.10), $n \to n-1$ in the latter equation.

6. MISCELLANEOUS REMARKS

Determinantal Values

Computation gives us the pleasing and somewhat unexpected result,

$$\begin{vmatrix} M_n(x) & M_{n+1}(x) & M_{n+2}(x) \\ M_{n+1}(x) & M_{n+2}(x) & M_{n+3}(x) \\ M_{n+2}(x) & M_{n+3}(x) & M_{n+4}(x) \end{vmatrix} = (-1)^n, \tag{6.1}$$

which is clearly independent of x. Establishing (6.1) requires (2.1) and the Simson formula for $P_n(x)$ [4, (3.30)], together with some routine determinantal manipulations.

Similarly, the appropriate algebraic maneuvering leads to

$$\begin{vmatrix} N_n(x) & N_{n+1}(x) & N_{n+2}(x) \\ N_{n+1}(x) & N_{n+2}(x) & N_{n+3}(x) \\ N_{n+2}(x) & N_{n+3}(x) & N_{n+4}(x) \end{vmatrix} = 8(-1)^{n+1}\Delta^2, \tag{6.2}$$

which is **not** independent of x [cf. (6.1)].

An investigation into a corresponding determinantal value for $\{\mathcal{Q}_n(x)\}$ led to some unlovely algebra which was abandoned. However, to compensate for this disappointment, our general endeavors are rewarded with a result such as (6.1).

Diagonal Functions

When analyzing the structure of a set of polynomials, it is sometimes instructive to consider the rising (and descending) diagonal functions which, in the inward eye, are inherent in the system along upward (downward) slanting "lines." See, for instance, [1].

While such new polynomial sets can create some interest (e.g., the existence of certain differential equations—partial or ordinary), preliminary efforts with polynomials exposed in this paper do not seem particularly promising. But for "Time's winged chariot hurrying near," one could be encouraged to persevere with this challenge.

Other MinMax Systems

MinMax numbers for the Fibonacci numbers are exhibited in [2]. From these one may construct corresponding Fibonacci polynomials. Likewise, for the Lucas numbers and their polynomials. Experience suggests that an interconnected theory for these polynomials and for Lucas polynomials analogous to that established in the preceding treatment might be possible.

One does not have to be psychic to expect that similar developments might be worthwhile involving polynomials abstracted from other number sequences, e.g., Jacobsthal numbers.

The Tables

Of passing aesthetic appreciation is the varying pattern of constants in the polynomials listed in Tables 1-5, e.g., in Table 3 the sequence $\{1, 1, 3, 3, 5, 5, 7, 7, 9, 9, \ldots\}$ for $\{N_n(x)\}$.

7. CONCLUSION

It should be noted that numerical, i.e., nonpolynomial, recurrences specialized from (2.1), (3.2), (4.1), and (5.2)—along with other recurrences with a fixed additive constant—have recently been investigated in [2].

One wonders, *en passant*, what opportunities for discovery might exist from the innovative invention of polynomial sequences of the kind defined by $p_n(x) = xq_n(x) + q_{n-1}(x)$, or perhaps $p_n^*(x) = xq_n^*(x) + q_{n-1}^*(x)$.

There are further possible variations on our theme. Among these is the extension of our polynomials to negatively-subscripted symbols, e.g., $\{M_{-n}(x)\}$, but our ambitions are tempered by the sobering reminder of Longfellow that "Art is long and Time is fleeting."

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Announcement

SEVENTH INTERNATIONAL CONFERENCE ON FIBONACCI NUMBERS AND THEIR APPLICATIONS

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The SEVENTH INTERNATIONAL CONFERENCE ON FIBONACCI NUMBERS AND THEIR APPLICATIONS will take place at Technische Universität Graz from July 14 to July 19, 1996. This conference will be sponsored jointly by the Fibonacci Association and Technische Universität Graz.

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