ON k-SELF-NUMBERS AND UNIVERSAL GENERATED NUMBERS*

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1. INTRODUCTION

In 1963, D. R. Kaprekar [1] introduced the concept of self-numbers. Let $k>1$ be an arbitrary integer. A natural number m is said to be a k-self-number iff the equation

$$m=n+d_k(n)$$

has no solution in an integer $n>0$, where $d_k(n)$ denotes the sum of digits of n while represented in the base k. Otherwise, we say that m is a k-generated number. And m is said to be a universal generated number if it is generated in every base. For example, 2, 10, 14, 22, 38, etc. are universal generated numbers. The number 12 is 4-generated by 9, but it is a 6-self-number.

In 1973, V. S. Joshi [2] proved that "if k is odd, then m is a k-self-number iff m is odd," i.e., every even number in an odd base is a generated number.

In 1991, R. B. Patel ([3], M.R. 93b:11011) tested for self-numbers in an even base k. What he proved is: $2ki$, $4k+2$, k^2+2k+1 are k-self-numbers in every even base $k\geq 4$.

In the present paper, we first prove some new results on self-numbers in an even base k.

Theorem 1: Suppose

$$m=b_0+b_1k,\ 0\leq b_0<k,\ 0<b_1<k,\ 2|k,\ k\geq 4.$$

Then m is a k-self-number iff $b_0-b_1=-2$.

In particular, $2k$, $3k+1$, $4k+2$, $5k+3$, etc. are k-self-numbers.

Theorem 2: Suppose

$$m=b_0+b_1k+b_2k^2,\ 0\leq b_0<k,\ 0\leq b_1<k,\ 0<b_2<k,\ 2|k,\ k\geq 4.$$

Then m is a k-self-number iff b_0, b_1, and b_2 satisfy one of the following conditions: $b_1=0$, $b_0-b_1-b_2=-4$ or $k-3$; $b_1=1$, $b_0-b_1-b_2=-2$ or -4; $b_1=2$ or 3, $b_0-b_1-b_2=-2$; $b_1\geq 4$, $b_0-b_1-b_2=-2$ or $-k-3$.

In particular, k^2+k, k^2+2k+1, k^2+3k+2, $2k^2+k+1$, $2k^2+2k+2$, $3k^2+k+2$, $5k^2+1$ $(k\geq 6)$, $4k^2+k+1$ $(k\geq 6)$, $5k^2-k$ $(k\geq 6)$, k^3-k^2+4k, etc. are k self-numbers.

Secondly, we study the number $G(x)$ of universal generated numbers $m\leq x$. It is not known if $G(x)\to\infty$ but, as an ingenious application of Theorem 1, we prove that $G(x)\leq 2\sqrt{x}$. As a matter of fact, we obtain

Theorem 3: Every universal generated number can be represented in only one way, in the form $2^sn+2^{s-1}-2$, with $s\geq 3$, $n\leq 2^{s-2}$. Moreover, for all $x>1$, one has $G(x)\leq 2\sqrt{x}$.

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2. PROOF OF THEOREM 1

If possible, let m be k-generated by some n, where

$$n=\sum_{i=0}^{t} a_i k^i,\quad 0\le a_i<k,\ 0\le i\le t.$$

Then

$$d_k(n)=\sum_{i=0}^{t} a_i \quad\text{and}\quad m=n+d_k(n)=\sum_{i=0}^{t} a_i(k_i+1).$$

Since $m=b_0+b_1k<k+(k-1)k=k^2$, we have $a_i=0$ for $i\ge 2$, i.e.,

$$b_0+b_1k=2a_0+a_1(k+1),\quad 0\le a_0,\ a_1<k. \tag{1}$$

Here $a_1>b_1$ or $a_1<b_1-2$ is impossible, so that $a_1=b_1-i,\ 0\le i\le 2$.

(A) If $i=0$, then (1) holds iff $b_0-b_1\ge 0$ is even;

(B) If $i=1$, then (1) holds iff $b_0-b_1\le k-3$ is odd;

(C) If $i=2$, then (1) holds iff $b_0-b_1\le -4$ is even.

Hence, m is a k-self-number iff $b_0-b_1=-2$ or $k-1$. The latter is impossible because $b_0\le k-1$. This completes the proof of Theorem 1.

3. PROOF OF THEOREM 2

If possible, let m be k-generated by some n. As in the proof of Theorem 1, we have

$$b_0+b_1k+b_2k^2=2a_0+a_1(k+1)+a_2(k^2+1), \tag{2}$$

with $b_2-1\le a_2\le b_2$.

Case I. $a_2=b_2$. From (2), we see that $a_1\le b_1$. Taking $a_1=b_1-j,\ j\ge 0$, we have

$$b_0-b_1-b_2+j(k+1)=2a_0. \tag{3}$$

Noting that $0\le a_0<k$, one has:

(A) If $j=0$, then (3) holds iff $b_0-b_1-b_2\ge 0$ is even;

(B) If $j=1$, then (3) holds iff $b_0-b_1-b_2\ge -k-1$ is odd and $b_1\ge 1$;

(C) If $j=2$, then (3) holds iff $b_0-b_1-b_2\le -4$ is even and $b_1\ge 2$;

(D) If $j=3$, then (3) holds iff $b_0-b_1-b_2\le k-5$ is odd and $b_1\ge 3$;

(E) If $j\ge 4$, then (3) never holds.

Case II. $a_2=b_2-1$. Taking $a_1=k-j,\ j\ge 1$, it follows from (2) that

$$(b_1+j-1)k=2a_0-j-1+b_2-b_0$$

or

$$b_0-b_2+(b_1+j-1)k+j-1=2a_0. \tag{4}$$

Since $2a_0-j-1+b_2-b_0\le 3(k-1)$, one has $b_1+j-1\le 2$. Noting that $0\le a_0\le k-1$, one has:

(A)′ If $b_1=0,\ j=1$, then (4) holds iff $b_0-b_2\ge -2$ is even;

(B)′ if $b_1=0,\ j=2$, then (4) holds iff $b_0-b_2\le k-5$ is odd;

(C)′ If $b_1 = 0$, $j = 3$, then (4) holds iff $b_0 - b_2 \leq -6$ is even;

(D)′ If $b_1 = 1$, $j = 1$, then (4) holds iff $b_0 - b_2 \leq k - 4$ is even;

(E)′ If $b_1 = 1$, $j = 2$, then (4) holds iff $b_0 - b_2 \leq -5$ is odd;

(F) If $b_1 = 2$, $j = 1$, then (4) holds iff $b_0 - b_2 \leq -4$ is even;

(G) If $b_1 \geq 3$, then (4) never holds.

Thus, (A)′, (B)′, and (C)′ together imply that if $b_1 = 0$, (4) does not hold iff $b_0 - b_2 = -4$ or $k - 3$, i.e., $b_0 - b_1 - b_2 = -4$ or $k - 3$. According to Case I, (2) has no solution iff $b_0 - b_1 - b_2 = -4$ or $k - 3$.

If $b_1 = 1$, (D)′ and (E)′ together imply that (4) does not hold iff $b_0 - b_2 > k - 4$ or $k - 4 > b_0 - b_2 > -5$ is odd, i.e., $b_0 - b_1 - b_2 > k$ or $k - 5 > b_0 - b_1 - b_2 > -6$ is even. According to Case I, (2) has no solution iff $b_0 - b_1 - b_2 = -2$ or -4.

If $b_1 = 2$, then from (F) (4) does not hold iff $b_0 - b_2 > -4$ or is odd, i.e., $b_0 - b_1 - b_2 > -6$ or is odd. According to Case I, (2) has no solution iff $b_0 - b_1 - b_2 = -2$.

If $b_1 \geq 3$, (4) never holds. According to Case I, (2) has no solution iff $b_0 - b_1 - b_2 = -2$ or $-k - 3$. For the latter, $b_1 \geq 4$. This completes the proof of Theorem 2.

4. PROOF OF THEOREM 3

Let $f_s(n)$ denote $2^s n + 2^{s-1} - 2$, where $s \geq 1$ and $n \geq 1$. Then $f_1(n) = 2n - 1$, $f_2(n) = 4n$, $f_3(n) = 8n + 2$, $f_4(n) = 16n + 6$, ... Noting that $f_s(n) = f_{s_1}(n_1)$ iff $n - n_1$, $s = s_1$, one has from the fundamental theorem of arithmetic: every positive integer can be represented in only one way, in the form $2^s n + 2^{s-1} - 2$. If $s = 1$, $n \geq 2$, it is clear that $f_1(n) = 2n - 1$ is not generated by $2n$. If $s \geq 2$, taking $b_0 = 2^{s-1} - 2$, $b_1 = 2^{s-1}$, $k = 2n$, and applying Theorem 1 we see that $f_s(n)$ is a k-self-number, i.e., it is not a universal generated number if $n > 2^{s-2}$. Moreover,

$$G(x) \leq \sum_{\substack{1 \leq 2^s n + 2^{s-1} - 2 \leq x \\ s \geq 1,\, n \leq 2^{s-2}}} 1 \leq \sum_{s \geq 1} \min\{2^{s-2}, x/2^s\} \leq \sum_{s \leq (1/2)\log_2 x + 1} 2^{s-2} + \sum_{s > (1/2)\log_2 x + 1} x/2^s \leq 2\sqrt{x}.$$

This completes the proof of Theorem 3.

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