VAROL'S PERMUTATION AND ITS GENERALIZATION

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INTRODUCTION

Let S_n be the set of $n!$ permutations $\Pi = a_1a_2\dots a_n$ of $\mathbb{Z}_n = \{1, 2, \dots, n\}$. For $\Pi \in S_n$, we write $\Pi = a_1a_2\dots a_n$, where $\Pi(i) = a_i$.

Definition 1: $P_n := \{\Pi \in S_n \mid a_{i+1} \neq a_i + 1 \text{ for all } i, 1 \le i < n\}$, $|P_n| = p_n$;
$\overline{P}_n := \{\Pi \in P_n \mid a_n = n\}$, $|\overline{P}_n| = \overline{p}_n$;
$P'_n = \{\Pi \in S_n \mid a_{i+1} \neq a_i - 1 \text{ for all } i, 1 \le i < n\}$, $|P'_n| = p'_n$;
$\overline{P}'_n = \{\Pi \in P'_n \mid a_n = n\}$, $|\overline{P}'_n| = \overline{p}'_n$.

Definition 2: $$T_n := \left\{\Pi \in P_n \,\middle|\, \sum_{j=1}^{i} a_j > \frac{i(i+1)}{2} \text{ for any } i, 1 \le i < n\right\},$$
$T_1 = \phi$;
$$T'_n := \left\{\Pi \in P'_n \,\middle|\, \sum_{j=1}^{i} a_j > \frac{i(i+1)}{2} \text{ for any } i, 1 \le i < n\right\},$$
$T'_1 = \phi$, $|T_n| = t_n$, $|T'_n| = t'_n$.

Definition 3: $R_n := P_n \cap P'_n$, $G_n := T_n \cap T'_n$, $|G_n| = r_n$, $|G_n| = g_n$.

From Computer Science, Varol first studied T_n and obtained the recurrence for t_n (see [1]). In [2], R. Luan discussed the enumeration of T'_n and G_n. This paper deals with the above problems in a way that is different from [1] and [2]. A series of new formulas of enumeration for t_n, t'_n, and g_n (Theorems 1-9) has been derived.

1. ENUMERATION OF t_n AND t'_n

Lemma 1:
$$P_n = (n-1)!\sum_{k=0}^{n-1}(-1)^k\frac{n-k}{k!} = D_n + D_{n-1}, \tag{1.1}$$
where $D_n = n!\sum_{j=0}^{n}\frac{(-1)^j}{j!}$ is the number of derangement of $\{1, 2, \dots, n\}$ (see [3]).

Proof: Consider the set $S' = \{(1, 2); (2, 3); \dots; (n-1, n)\}$. We say that an element $(j, j+1)$ of S' is in a permutation Π if $\Pi(i) = j$, $\Pi(i+1) = j+1$ for some i.

Let W_k be the number of permutations in S_n containing at least k elements of S'. The number of ways of taking k elements from S' is $\binom{n-1}{k}$. Suppose the k elements have j digits in common. Then these k elements form $(k-j)$ continuous sequences of natural numbers, each of which is called a *block*. Thus, the number of remaining elements in $\mathbb{Z}_n$ is $(n-2k+j)$. The number of permutations of $(n-2k+j)$ elements and $(k-j)$ blocks is $[(n-2k+j)+(k-j)]! = (n-k)!$. Hence, $W_k = \binom{n-1}{k}(n-k)!$.

By the principle of inclusion and exclusion (see [3]):

$$P_n = \sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}(n-k)! = (n-1)!\sum_{k=0}^{n}(-1)^k\frac{n-k}{k!} = D_n + D_{n-1},$$

where $D_n = n!\sum_{k=0}^{n}\frac{(-1)^k}{k!}$ is the number of derangement of $\{1, 2, \ldots, n\}$ (see [3], p. 59). □

Lemma 2:

$$\overline{p_n} = \sum_{j=0}^{n-1}(-1)^{n+1-j}p_j, \text{ where } p_0 = 1. \tag{1.2}$$

Proof: It is easy to see that $\overline{p_n} = p_{n-1} - \overline{p_{n-1}}$. Applying the above recurrence repeatedly, we get (1.2). □

Let

$$P(x) = \sum_{n=0}^{\infty} p_n x^n. \tag{1.3}$$

Theorem 1:

$$t_n = \sum_{i=0}^{n}(-1)^{n-i}p_i - \sum_{i=1}^{n-2}p_i t_{n-i}. \tag{1.4}$$

Proof: Consider the following subset P_n^0 of P_n.

$$\begin{aligned} P_n^0 = \Big\{&(a_1a_2\ldots a_i\ldots a_n) \in P_n \mid \text{for some } i,\ 1 \le i < n,\\ &a_1 + a_2 + \cdots + a_i = \frac{i(i+1)}{2}, \text{ but for } i < j < n,\\ &a_1 + a_2 + \cdots + a_j > \frac{j(j+1)}{2}\Big\}. \end{aligned}$$

If $i < n-1$, then the number of such permutations is $p_i t_{n-i}$; if $i = n-1$, the number is $\overline{p_{n-1}}$. Thus,

$$|P_n^0| = \sum_{i=1}^{n-2}p_i t_{n-i} + \overline{p_{n-1}}.$$

Hence,

$$t_n = p_n - |P_n^0| = p_n - \sum_{i=1}^{n-2}p_i t_{n-i} - \overline{p_{n-1}}.$$

Substituting (1.2) into the above formula, we have (1.4). □

To simplify (1.4), we establish a lemma as follows.

Lemma 3: If

$$t_n = a_n - \sum_{i=1}^{n-2}b_i t_{n-i}, \quad n \ge 2,$$

then

$$t_n = \begin{vmatrix} 1 & & & & & a_2 \\ b_1 & 1 & & \mathbf{0} & & a_3 \\ b_2 & b_1 & 1 & & & a_4 \\ \cdots & \cdots & \ddots & & & \cdots \\ \cdots & \cdots & & b_1 & 1 & \cdots \\ b_{n-2} & b_{n-3} & \cdots & \cdots & b_1 & a_n \end{vmatrix}. \tag{1.5}$$

Proof: This follows from the expansion of the determinant along the bottom row. □

Hence, we can write (1.4) as

Theorem 2:

$$t_n = \begin{vmatrix} 1 & & & & & p_2 - p_1 + 1 \\ p_1 1 & & & \mathbf{0} & & p_3 - p_2 + p_1 - 1 \\ p_2 & p_1 & 1 & & & p_4 - p_3 + p_2 - p_1 + 1 \\ \cdots & \cdots & \ddots & & & \cdots \\ \cdots & \cdots & & p_1 & 1 & \cdots \\ p_{n-2} & p_{n-3} & \cdots & \cdots & p_1 & p_n - p_{n-1} + p_{n-2} - \cdots + (-1)^n \end{vmatrix}, \quad n \geq 2, \tag{1.6}$$

Example 1:

$$t_5 = \begin{vmatrix} 1 & 0 & 0 & p_2 - p_1 + 1 \\ p_1 & 1 & 0 & p_3 - p_2 + p_1 - 1 \\ p_2 & p_1 & 1 & p_4 - p_3 + p_2 - p_1 + 1 \\ p_3 & p_2 & p_1 & p_5 - p_4 + p_3 - p_2 + p_1 - 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 2 \\ 1 & 1 & 1 & 9 \\ 3 & 1 & 1 & 44 \end{vmatrix} = 33.$$

Let $T(x) = \sum_{n=0}^{\infty} t_n x^n$, $t_0 = 0$. Then we have

Theorem 3:

$$T(x) = (1+x)^{-1} - \frac{1}{P(x)}, \text{ where } P(x) = \sum_{n=0}^{\infty} p_n x^n. \tag{1.7}$$

Proof: From (1.4),

$$\begin{aligned} P(x)T(x) &= \sum_{n=0}^{\infty} p_n x^n \sum_{n=0}^{\infty} t_n x^n = \sum_{n=0}^{\infty} \left(\sum_{i=0}^{\infty} p_i t_{n-i} \right) x^n \\ &= \sum_{n=0}^{\infty} \left(\sum_{i=0}^{\infty} (-1)^{n-i} p_i \right) x^n = \sum_{m=0}^{\infty} (-1)^m x^m \sum_{n=0}^{\infty} p_n x^n - 1 = (1+x)^{-1} P(x) - 1. \end{aligned}$$

Hence,

$$T(x) = (1+x)^{-1} - \frac{1}{P(x)}. \quad \square$$

Now we shall consider t_n'.

Lemma 4:

$$p_n' = p_n. \tag{1.8}$$

Proof: If $(a_1 a_2 \ldots a_n) \in P_n$, then $(a_n a_{n-1} \ldots a_2 a_1) \in P_n'$. The above correspondence is one-to-one; thus, $|P_n| = |P_n'|$. □

Arguing as in the proof of Theorem 1, we get

Theorem 4:

$$t'_n = p_n - \sum_{i=1}^{n-1} p_i t'_{n-i}. \tag{1.9}$$

By recurrence (1.9) and Lemma 4, we have an explicit formula for t'_n as follows.

Theorem 5:

$$t_n = \begin{vmatrix} 1 & & & & & p_1 \\ p_1 & 1 & & 0 & & p_2 \\ p_2 & p_1 & 1 & & & p_3 \\ \cdots & \cdots & & \ddots & & \cdots \\ \cdots & \cdots & & & 1 & p_{n-1} \\ p_{n-1} & p_{n-2} & \cdots & \cdots & p_1 & p_n \end{vmatrix}. \tag{1.10}$$

Let the generating function for t'_n be $T'(x) = \sum_{n=0}^{\infty} t'_n x^n$, $t'_n = 0$.

Theorem 6:

$$T'(x) = 1 - \frac{1}{P(x)}. \tag{1.11}$$

Proof: By (1.9), $\sum_{i=0}^{n} p_i t'_{n-i} = p_n, n \geq 1$. Thus, $P(x)T'(x) = P(x) - 1$, and we have

$$T'(x) = 1 - \frac{1}{P(x)}, \text{ as required. } \square$$

Lemma 5:

$$t_n = t'_n + (-1)^n. \tag{1.12}$$

Proof: Since $T(x) = 1/(1+x) - 1/[P(x)]$, $T'(x) = 1 - 1/[P(x)]$; hence,

$$T(x) - T'(x) = -\frac{x}{1+x},$$

i.e.,

$$\sum_{n=0}^{\infty} (t_n - t'_n) x^n = \sum_{n=0}^{\infty} (-1)^n x^n.$$

Comparing the coefficients of x^n, we have $t_n - t'_n = (-1)^n$, i.e., (1.12). □

According to (1.12) and (1.10), we have a simple expression for t_n as follows:

Theorem 7:

$$t_n = \begin{vmatrix} 1 & & & & & p_1 \\ p_1 & 1 & & 0 & & p_2 \\ p_2 & p_1 & 1 & & & p_3 \\ \cdots & \cdots & \cdots & \ddots & & \cdots \\ \cdots & \cdots & \cdots & & 1 & p_{n-1} \\ p_{n-1} & \cdots & \cdots & p_2 & p_1 & p_n \end{vmatrix} + (-1)^n. \tag{1.13}$$

For Example 1, we can count t_5 by the above formula:

$$t_5 = \begin{vmatrix} 1 & 0 & 0 & 0 & p_1 \\ p_1 & 1 & 0 & 0 & p_2 \\ p_2 & p_1 & 1 & 0 & p_3 \\ p_3 & p_2 & p_1 & 1 & p_4 \\ p_4 & p_3 & p_2 & p_1 & p_5 \end{vmatrix} + (-1)^5 = \begin{vmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 3 \\ 3 & 1 & 1 & 1 & 11 \\ 11 & 3 & 1 & 1 & 53 \end{vmatrix} - 1 = 33.$$

Since the enumeration for p_n is simple (we may look it up in a table of values of D_n) counting t_n by (1.1) and (1.13) is easier than the method of [2].

2. ENUMERATION OF g_n

Definition 4: $R_{n,i} := \{\Pi = (a_1 a_2 \ldots a_n) \in S_n \mid \Pi$ contains either $(i, i+1)$ or $(i+1, i)$, but for $\forall j < i$, Π contains neither $(j, j+1)$ nor $(j+1, j)\}$. Let $|R_{n,i}| = r_{n,i}$.

Lemma 6: $r_{n,i} = 2(n-1)!$, $n \geq 2$; $r_{1,1} = 1$.

Proof: By definition, $R_{n,1}$ is all permutations of S_n containing either $(1, 2)$ or $(2, 1)$. If we regard $(1, 2)$ as an element, then $(1, 2)$ and the remaining $(n-2)$ elements of $\mathbb{Z}_n$ form $(n-1)!$ permutations. Note that there are two permutations of $\{1, 2\}$. Thus, $r_{n,1} = 2(n-1)!$. □

Lemma 7: $r_{n,2} = 2(n-1)! - 2(n-2)!$.

Proof: Let us count the number of permutations containing either $(2, 3)$ or $(3, 2)$ but neither $(1, 2)$ nor $(2, 1)$.

Arguing as in Lemma 6, we know that the number of permutations containing either $(2, 3)$ or $(3, 2)$ is $2(n-1)!$. We have to eliminate those permutations containing $(1, 2, 3)$ or $(3, 2, 1)$, the number of which is $2(n-2)!$ by an argument analogous to Lemma 6. Thus, we have proved Lemma 7. □

Lemma 8:
$$r_{n,i} - r_{n,i-1} - r_{n-1,i-1} - r_{n-1,i-2}, \text{ where } 2 \leq i < n,\ r_{n,0} = 0. \tag{2.1}$$

Proof: Each permutation of $R_{n,i}$ contains $(i, i+1)$ or $(i+1, i)$. If $(i, i+1)$ is followed by $i-1$ or if $(i+1, i)$ is preceded by $i-1$, then $(i+1)$ is removed, and we subtract 1 from every digit that is greater than $i+1$. Thus, we get an element of $R_{n-1,i-1}$. Conversely, given a permutation of $R_{n-1,i-1}$, we add 1 to each element greater than i and then interpose an $i+1$ between $(i, i-1)$ or $(i-1, i)$. This yields an element of $R_{n,i}$.

If $(i, i+1)$ is not followed by $i-1$ or preceded by $(i+1, i)$, we regard $(i, i+1)$ or $(i+1, i)$ as a single element and subtract 1 from every digit greater than $i+1$. This yields an element $S_{n-1} - R_{n-1,1} \cup R_{n-1,2} \cup \cdots \cup R_{n-1,i-1}$. Thus,

$$r_{n,i} = r_{n-1,i-1} + 2\left[(n-1)! - \sum_{j=1}^{i-1} r_{n-1,j}\right].$$

Hence,

$$r_{n,i} = 2\left[(n-1)! - \sum_{j=1}^{i-2} r_{n-1,j}\right] - r_{n-1,i-1}, \tag{2.2}$$

i.e.,

$$r_{n,i} = (r_{n,i-1} - r_{n-1,i-2}) - r_{n-1,i-1}. \quad \square$$

Using Lemmas 6, 7, and 8 above, we can express $r_{n,k}$ in terms of $r_{n,1}$:

$$r_{n,2} = r_{n,1} - r_{n-1,1} = 2(n-1)! - 2(n-2)!,$$
$$r_{n,3} = r_{n,1} - 3r_{n-1,1} + r_{n-2,1} = 2(n-1)! - 6(n-2)! + 2(n-3)!,$$
$$r_{n,4} = r_{n,1} - 5r_{n-1,1} + 5r_{n-2,1} - r_{n-3,1} = 2(n-1)! - 10(n-2)! + 10(n-3)! - 2(n-4)!,$$
$$\dots$$

In general, let

$$r_{n,k} = a_{k,1}2(n-1)! - a_{k,2}2(n-2)! + \cdots + a_{k,k}(-1)^{k+1}2(n-k)!.$$

Obviously $a_{k,i}$ is independent of n. It only depends on k. We can prove

Lemma 9:

$$a_{k,1} = 1,$$
$$a_{k,j} = a_{k-1,j} + a_{k-1,j-1} + a_{k-2,j-1}, \quad 1 < j < k, \tag{2.3}$$
$$a_{k,k} = 1.$$

Proof: Since $r_{n,k} = \sum_{j=1}^{k} a_{k,j}2(n-j)!(-1)^{j+1}$ by (2.1), we have:

$$r_{n,k} = r_{n,k-1} - r_{n-1,k-1} - r_{n-1,k-2}$$
$$= \sum_{j=1}^{k-1}(-1)^{j+1}a_{k-1,j}2(n-j)! - \sum_{j=1}^{k-1}(-1)^{j+1}a_{k-1,j}2(n-1-j)! - \sum_{j=1}^{k-2}(-1)^{j+1}a_{k-2,j}2(n-1-j)!$$
$$= a_{k-1,1}2(n-1)! + \sum_{j=2}^{k-1}(-1)^{j+1}(a_{k-1,j} + a_{k-1,j-1} + a_{k-2,j-1})2(n-j)! + (-1)^{k+1}a_{k-1,k-1}2(n-k)!.$$

Comparing the two formulas above, we obtain relations for $a_{n,k}$ as follows:

$$a_{k,1} = a_{k-1,1}, \text{ thus, } a_{k,1} = a_{k-1,1} = a_{k-2,1} = \cdots = a_{1,1} = 1,$$
$$a_{k,j} = a_{k-1,j} + a_{k-1,j-1} + a_{k-2,j-1}, \quad 1 < j < k,$$
$$a_{k,k} = a_{k-1,k-1}, \text{ thus, } a_{k,k} = a_{k-1,k-1} = \cdots = a_{1,1} = 1. \quad \square$$

Lemma 10:

$$a_{k,j} = a_{k,k+1-j}. \tag{2.4}$$

Proof: We prove the lemma by induction on k. For $k = 1, 2$, or 3, this is straightforward. Suppose that (2.3) holds for $k-1$. By (2.2),

$$a_{k,j} = a_{k-1,j} + a_{k-1,j-1} + a_{k-2,j-1} = a_{k-1,k-j} + a_{k-1,k-j+1} + a_{k-2,k-j} = a_{k,k+1-j}.$$

By (2.3) and (2.4), we easily obtain the expression for $r_{n,k}$:

$$r_{n,1} = 2(n-1)!,$$
$$r_{n,2} = 2(n-1)! - 3\cdot 2(n-2)! + 2(n-3)!,$$
$$r_{n,3} = 2(n-1)! - 5\cdot 2(n-2)! + 5\cdot 2(n-3)! - 2(n-4)!,$$
$$r_{n,4} = 2(n-1)! - 7\cdot 2(n-2)! + 13\cdot 2(n-3)! - 7\cdot 2(n-4)! + 2(n-5)!.$$

For $a_{k,j}$, using

$$\begin{array}{c} a_{k-2,j-1} \\ + \\ a_{k-1,j-1} + a_{k-1,j} \\ \qquad\quad \| \\ \qquad\quad a_{k,j}, \end{array}$$

we obtain the above formulas one by one. Now, using (2.1), we get the table for $r_{n,k}$ shown below.

TABLE 1. $r_{n,k}$ **(** $k < n$ **)**, $r_{n,n} := r_n$

n \ k	0	1	2	3	4	5	6
0	0						
1	0	1					
2	0	2	0				
3	0	4	2	0			
4	0	12	8	2	2		
5	0	48	36	16	6	14	
6	0	240	192	108	56	34	90
7	0	1440	1200	768	468	304	214

Setting $f_n(x) = \sum_{k=0}^{n-1} r_{n,k}x^k$, and letting $i = n-1$ in (2.2), we have

Corollary:

$$r_{n,n-1} = 2(n-1)! + r_{n-1,n-2} - 2f_{n-1}(1). \tag{2.5}$$

Lemma 11:

$$r_n = \tfrac{1}{2}(r_{n+1,n} - r_{n,n-1}). \tag{2.6}$$

Proof: Denote the set of permutations containing neither $(n-1, n+1, n)$ nor $(n, n+1, n-1)$ in $R_{n+1,n}$ by $R^*_{n+1,n} \subset R_{n+1,n}$.

For any $\alpha \in R_n$, inserting $n+1$ to the left or right of n in α, we get $\alpha' \in R^*_{n+1,n}$. Conversely, if $\alpha' \in R^*_{n+1,n}$, then eliminating $n+1$ yields $\alpha \in R_n$. Hence, $2r_n = |R^*_{n+1,n}|$.

Now we count $|R^*_{n+1,n}|$. It is sufficient to subtract the number of permutations containing $(n-1, n+1, n)$ or $(n, n+1, n-1)$ in $R_{n+1,n}$ from $r_{n+1,n}$. Regard $(n+1, n)$ as a single element. Then $\frac{1}{2}r_{n,n-1}$ is the number of permutations containing no $(n-1, n+1, n)$.

By a similar argument, the number of permutations containing no $(n, n+1, n-1)$ is $\frac{1}{2}r_{n,n-1}$. Thus, $|R^*_{n+1,n}| = r_{n+1,n} - \frac{1}{2}r_{n,n-1} - \frac{1}{2}r_{n,n-1}$. Since $2r_n = |R^*_{n+1,n}|$, we get (2.6). □

Lemma 12:

$$r_n = (n-1)(n-1)! + f_{n-1}(1) - f_n(1) + r_{n-1}, \tag{2.7}$$

$$r_n = \sum_{i=1}^{n} (n-i)(n-i)! - f_n(1), \tag{2.8}$$

where $0 \cdot 0! = 1$.

Proof: Substituting (2.5) into (2.6), we obtain

$$r_n = \tfrac{1}{2}[2\cdot n! + r_{n,\,n-1} - 2f_n(1) - 2(n-1)! - r_{n-1,\,n-2} + 2f_{n-1}(1)].$$

Using (2.6), we get (2.7). Applying (2.7) repeatedly, we have

$$r_n = \sum_{i=1}^{n-2}(n-i)(n-i)! - f_n(1) + r_2 + f_2(1).$$

Since $r_2 = 0$ and $f_n(1) = 2$, we obtain (2.8). □

By (2.6), it is easy to get r_n from $r_{n,\,k}$. In Table 1 we denote $r_{n,\,n} = r_n$. Thus, by (2.6), the values of r_n on the principal diagonal can be obtained as half the difference between the two adjacent elements on the secondary diagonal. Unfortunately, we cannot count r_n until we complete Table 1. But (2.7) and (2.8) can do that, namely, both $r_{n,\,k}$ and r_n are counted without $r_{n+1,\,k}$.

If we set $f_{n+1}(x) = \sum_{k=0}^{n} r_{n+1,\,k}x^k$, we have

Lemma 13:

$$f_n(x) = \frac{x}{1-x}[2(n-1)! - 2r_{n-1}x^{n-1} - (1+x)f_{n-1}(x)]. \tag{2.9}$$

Proof: By (2.1), we have

$$\sum_{k=2}^{n-1} r_{n,\,k}x^k = \sum_{k=2}^{n-1} r_{n,\,k-1}x^k - \sum_{k=2}^{n-1} r_{n-1,\,k-1}x^k - \sum_{k=2}^{n-1} r_{n-1,\,k-2}x^k,$$

$$f_n(x) - r_{n,\,1}x = xf_n(x) - r_{n,\,n-1}x^n - xf_{n-1}(x) - x^2 f_{n-1}(x) + r_{n-1,\,n-2}x^n.$$

By (2.6), we have

$$(1-x)f_n(x) = 2(n-1)!\,x - 2r_{n-1}x^n - x(1+x)f_{n-1}(x). \quad \square$$

Example 2: Since $f_4(x) = 12x + 8x^2 + 2x^3$, $r_4 = 2$. We get

$$\begin{aligned} f_5(x) &= \frac{x}{1-x}[2\cdot 4! - 2\cdot 2x^4 - (1+x)(12+8x^2+2x^3)] \\ &= \frac{x}{1-x}[48 - 12 - 20x^2 - 10x^3 - 6x^4] \\ &= 48x + 36x^2 + 16x^3 + 6x^4, \end{aligned}$$

i.e., $r_{5,1} = 48$, $r_{5,2} = 36$, $r_{5,3} = 16$, $r_{5,4} = 6$. From (2.9), we may obtain

$$\begin{aligned} f_n(x) = \frac{2}{(1-x)^{n-2}}\Bigg[&\sum_{i=1}^{n-2}(-1)^{i-1}(n-i)!\,x^i(1+x)^{i-1}(1-x)^{n-i-2} \\ &+ x^n\sum_{i=1}^{n-2}(-1)^i r_{n-i}(1+x)^{i-1}(1-x)^{n-i-2} + (-1)^n x^{n-1}(1+x)^{n-2}\Bigg]. \end{aligned} \tag{2.10}$$

The application of (2.10) is not as convenient as that of (2.9), but it provides the following information: $r_{n,\,k}$ must be even. It coincides with the expression of $r_{n,\,k}$, i.e.,

$$r_{n,k} = \sum_{j=1}^{n} a_{k,j} \cdot 2(n-j)!(-1)^{j+1}.$$

Theorem 7:

$$g_n = \sum_{j=1}^{n} (-1)^{n-j} r_j - \sum_{j=1}^{n-1} r_j g_{n-j} + (-1)^n. \tag{2.11}$$

Proof: By a method similar to Theorem 1, it is easy to show that

$$g_n = r_n - \sum_{j=1}^{n-1} r_j g_{n-j} - \overline{r_{n-1}}, \tag{2.12}$$

where $\overline{r_{n-1}}$ is the number of permutations in R_{n-1} whose right-most entry is $n-1$.

Similar to Theorem 1, we have

$$\overline{r_{n-1}} = \sum_{j=1}^{n-1} (-1)^{n-1-j} r_j + (-1)^{n+1}. \tag{2.13}$$

Now, substituting (2.13) into (2.12), we obtain (2.11) as required. □

According to (2.11), we can count g_n by recurrence. Using (2.11) and noticing that $g_1 = g_2 = g_3 = 0$, we get an explicit formula for g_n.

Theorem 8:

$$g_n = \begin{vmatrix} 1 & & & & & r_4 \\ 1 & 1 & & & & r_5 - r_4 \\ r_2 & 1 & 1 & & & r_6 - r_5 + r_4 \\ \cdots & & \ddots & \ddots & & \cdots \\ \cdots & & & \ddots & 1 & r_{n-1} - \cdots + (-1)^{n-1} r_4 \\ r_{n-4} & r_{n-5} & \cdots & r_2 & 1 & r_n - r_{n-1} + \cdots + (-1)^n r_4 \end{vmatrix}, \quad n \geq 4. \tag{2.14}$$

Example 3:

$$g_6 = \begin{vmatrix} 1 & 0 & r_4 \\ 1 & 1 & r_5 - r_4 \\ r_2 & 1 & r_6 - r_5 + r_4 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 2 \\ 1 & 1 & 14-2 \\ 0 & 1 & 90-14+2 \end{vmatrix} = 68.$$

Let $G(x) = \sum_{n=0}^{\infty} g_n x^n$, $R(x) = \sum_{n=0}^{\infty} r_n x^n$, $r_0 = 0$.

Theorem 9: $G(x) = (1+x)^{-1} - (R(x)+1)^{-1}$.

Proof: By (2.11),

$$\sum_{j=1}^{n-1} r_j g_{n-j} + g_n = \sum_{j=1}^{n} (-1)^{n-j} r_j + (-1)^n.$$

Noticing that $g_1 = 0$, we have

$$G(x) \cdot R(x) + G(x) = (1+x)^{-1} R(x) + (1+x)^{-1} - 1;$$

thus, $G(x) = (1+x)^{-1} - (R(x)+1)^{-1}$. □

Corollary:

$$g_n = \frac{1}{n!}\frac{d^n}{dx^n}\left[(1+x)^{-1} - (R(x)+1)^{-1}\right]\Big|_{x=0}.$$

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