TRIANGULAR NUMBERS IN THE PELL SEQUENCE

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1. INTRODUCTION

In 1989, Luo Ming [5] proved Vern Hoggatt's conjecture that the only triangular numbers in the Fibonacci sequence $\{F_n\}$ are 1, 3, 21, and 55. It is our purpose, in this paper, to show that 1 is the only triangular number in the Pell sequence $\{P_n\}$.

Aside from the proof itself, Ming's unique contribution in his paper was his development and use of the value of the Jacobi symbol $(8F_{2kg+m}+1|L_{2k})$, where $\{L_n\}$ is the sequence of Lucas numbers, g is odd, and $k \equiv \pm 2 \pmod 6$. In other papers involving similar arguments, the value of the Jacobi symbol $(f(2kg+m)|L_t)$, for certain functions $f(n)$ of F_n and/or L_n, has often been obtained for t a divisor of k, but not for t equal to $2k$ (e.g., [1], [2], [3], and [7]).

It is immediate, from the definition, that an integer k is a triangular number iff $8k+1$ is a perfect square >1. We shall employ an argument similar to that used by Ming to show that, for every integer $n \neq \pm 1$, there exists an integer $w(n)$, such that $8P_n+1$ is a quadratic nonresidue modulo $w(n)$.

We require the sequence of "associated" Pell numbers defined by $Q_0 = 1$, $Q_1 = 1$ and, for all integers $n \geq 0$, $Q_{n+2} = 2Q_{n+1} + Q_n$. The first few Pell and associated Pell numbers are

$$\{P_n\} = \{0, 1, 2, 5, 12, \ldots\} \quad \text{and} \quad \{Q_n\} = \{1, 1, 3, 7, 17, \ldots\}.$$

2. SOME IDENTITIES AND PRELIMINARY LEMMAS

The following formulas and identities are well known. For all integers n and m,

$$P_{-m} = (-1)^{m+1}P_m \quad \text{and} \quad Q_{-m} = (-1)^m Q_m, \tag{1}$$

$$P_{m+n} = P_m P_{n+1} + P_{m-1}P_n, \tag{2}$$

$$P_{m+n} = 2P_m Q_n - (-1)^n P_{m-n}, \tag{3}$$

$$P_{2^t m} = P_m(2Q_m)(2Q_{2m})(2Q_{4m})\cdots(2Q_{2^{t-1}m}), \tag{4}$$

$$Q_m^2 = 2P_m^2 + (-1)^m, \tag{5}$$

$$Q_{2m} = 2Q_m^2 - (-1)^m. \tag{6}$$

If $d = \gcd(m, n)$, then

$$\begin{cases} \gcd(P_m, Q_n) = Q_d & \text{if } m/d \text{ is even,} \\ \gcd(P_m, Q_n) = 1 & \text{otherwise (see [4]).} \end{cases} \tag{7}$$

We note that (6) readily implies that, if $t > 1$, then $Q_{2^t} \equiv 1 \pmod 8$.

Lemma 1: Let $k = 2^t, t \geq 1, g > 0$ be odd and m be any integer. Then,

(i) $P_{2kg+m} \equiv -P_m \pmod{Q_k}$, and
(ii) $P_{2kg} \equiv \pm P_{2k} \pmod{Q_{2k}}$.

Proof: ***(i)*** is known [and can be easily proved using (1) and (3)]. If $n = 2kg = 2k(g-1)+2k$, then, using (3), ***(ii)*** readily follows from

$$P_n = 2P_{2k(g-1)}Q_{2k} - (-1)^{2k}P_{2k(g-2)} \equiv -P_{2k(g-2)} \pmod{Q_{2k}}.$$

Lemma 2: If $k = 2^t, t \geq 1$, then $(8P_{2k}+1|Q_{2k}) = (-8P_{2k}+1|Q_{2k})$.

Proof: We first observe that each Jacobi symbol is defined. Indeed, if $d = \gcd(8P_{2k}+1, Q_{2k})$ or $\gcd(-8P_{2k}+1, Q_{2k})$, then, using (5), d divides

$$(8P_{2k}+1)\cdot(-8P_{2k}+1) = 1-64P_{2k}^2 = 1-32(Q_{2k}^2-1) = 33-2Q_{2k}^2.$$

Hence, $d|33$. But $33|P_{12}$ which implies $d = 1$, since, by (7), $\gcd(P_{12}, Q_{2k}) = 1$.

To establish the lemma, note that

$$(8P_{2k}+1|Q_{2k})(-8P_{2k}+1|Q_{2k}) = (1-64P_{2k}^2|Q_{2k}) = (33|Q_{2k}) = (Q_{2k}|33).$$

Now, by (6), $Q_4 = 17$, $Q_8 = 2Q_4^2 - 1 = 2\cdot 17^2 - 1 \equiv 16$ (mod 33), and by induction, $Q_{2k} \equiv 16$ (mod 33) if $t \geq 2$. Hence, if $t \geq 1$, $(Q_{2k}|33) = +1$.

Lemma 3: If $k = 2^t, t \geq 2$, then $(8P_k + Q_k|33) = -1$.

Proof: $Q_2 = 3$, $Q_4 = 17 \equiv -16$ (mod 33), and as observed in the proof of Lemma 2, $Q_{2^j} \equiv 16$ (mod 33), if $j \geq 3$. Hence, by (4), if $t \geq 2$,

$$8P_k = 8P_2(2Q_2)(2Q_4)\cdots(2Q_{2^{t-1}}) \equiv 8\cdot 2\cdot 6\cdot(\pm 1) \equiv \pm 3 \pmod{33},$$

so, $8P_k + Q_k \equiv \pm 13$ or ± 19 (mod 33) and both $(\pm 13|33)$ and $(\pm 19|33) = -1$.

From a table of Pell numbers (e.g., [6], p. 59), we find that $P_{24} \equiv 0$ (mod 9) and $P_{25} \equiv 1$ (mod 9). Using (2),

$$P_{n+24} = P_nP_{25} + P_{n-1}P_{24} \equiv P_n \pmod 9,$$

and we have, immediately,

Lemma 4: If $n \equiv m$ (mod 24), then $P_n \equiv P_m$ (mod 9).

3. THE MAIN THEOREM

Theorem: The term P_n of the Pell sequence is a triangular number iff $n = \pm 1$.

Proof: If $n = \pm 1$, P_n is the triangular number 1. By (1), if n is an even negative integer, then $8P_n + 1$ is negative, and if n is odd, then $P_{-n} = P_n$; hence, it suffices to show that $8P_n + 1$ is not a square for $n > 1$. Let $n = 2kg + m$, $k = 2^t, t \geq 1, g \geq 1$ odd, and assume P_n is a triangular number. [Then $(8P_n + 1|N) = +1$ for all odd integers N.]

Case 1. n odd. Since $n \equiv \pm 1 \pmod 4$, $8P_n + 1 = 8P_{2kg\pm 1} + 1 \equiv -7 \pmod{Q_k}$, by Lemma 1(i) and (1). But it is readily shown, using (6), that $Q_k \equiv 3 \pmod 7$. Hence,

$$(8P_n + 1 | Q_k) = (-7 | Q_k) = (Q_k | 7) = (3 | 7) = -1,$$

a contradiction.

Case 2 (mod 4): $n \equiv 2$. It is easily seen that $\{P_n\}$ has period 6 modulo 7, and that, for n even, $(8P_n + 1 | 7) = +1$ only if $n \equiv 0 \pmod 6$. Hence, $n \equiv \pm 6 \pmod{24}$. By Lemma 4,

$$8P_n + 1 \equiv \pm 8P_6 + 1 \equiv 3 \text{ or } 8 \pmod 9.$$

But 3 and 8 are quadratic nonresidues of 9, so $8P_n + 1$ is not a square.

Case 3. $n \equiv 0 \pmod 4$. By Lemma 1(ii) and Lemma 2,

$$(8P_n + 1 | Q_{2k}) = (8P_{2k} + 1 | Q_{2k}).$$

If $k = 2 (t = 1)$, $(8P_{2k} + 1 | Q_{2k}) = (97 | 17) = -1$. Assume $t \geq 2$. Now,

$$8P_{2k} + 1 \equiv 8P_{2k} + (2Q_k^2 - Q_{2k}) \equiv 2Q_k(8P_k + Q_k) \pmod{Q_{2k}}.$$

Let $s_k = 8P_k + Q_k$ [note that $s_k \equiv 1 \pmod 8$]. Then, using properties (5) and (6),

$$\begin{aligned}(8P_{2k} + 1 | Q_{2k}) &= (Q_k | Q_{2k})(s_k | Q_{2k}) = (Q_{2k} | Q_k)(Q_{2k} | s_k) \\ &= (2Q_k^2 - 1 | Q_k)(2P_k^2 + Q_k^2 | s_k) = (+1)(2P_k^2 + (s_k - 8P_k)^2 | s_k) \\ &= (66P_k^2 | s_k) = (33 | s_k) = (s_k | 33) = (8P_k + Q_k | 33) = -1,\end{aligned}$$

by Lemma 3, and the proof is complete.

REFERENCES

1. J. A. Antoniadis. "Generalized Fibonacci Numbers and Some Diophantine Equations." *The Fibonacci Quarterly* **23.3** (1985):199-213.
2. Brother U. Alfred. "On Square Lucas Numbers." *The Fibonacci Quarterly* **2.1** (1964):11-12.
3. J. H. E. Cohn. "On Square Fibonacci Numbers." *J. London Math. Soc.* **39** (1964):537-41.
4. W. L. McDaniel. "The G.C.D. in Lucas Sequences and Lehmer Number Sequences." *The Fibonacci Quarterly* **29.1** (1991):24-29.
5. L. Ming. "On Triangular Fibonacci Numbers." *The Fibonacci Quarterly* **27.2** (1989):98-108.
6. P. Ribenboim. *The Book of Prime Number Records*. New York: Springer-Verlag, 1989.
7. P. Ribenboim. "Square Classes of Fibonacci and Lucas Numbers." *Port. Math.* **46** (1989): 159-75.

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