SELF-GENERATING PYTHAGOREAN QUADRUPLES AND N-TUPLES

Paul Oliverio
Jefferson High School, Los Angeles, CA 90011
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1. INTRODUCTION

In a rectangular solid, the length of an interior diagonal is determined by the formula

$$a^2+b^2+c^2=d^2, \tag{1}$$

where a, b, c are the dimensions of the solid and d is the diagonal.

When a, b, c, and d are integral, a **Pythagorean quadruple** is formed.

Mordell [1] developed a solution to this Diophantine equation using integer parameters (m, n, and p), where $m+n+p\equiv 1 \pmod 2$ and $(m,n,p)=1$. The formulas are:

$$\begin{aligned} a&=2mp \qquad & c&=p^2-(n^2+m^2),\\ b&=2np \qquad & d&=p^2+(n^2+m^2). \end{aligned} \tag{2}$$

However, the Pythagorean quadruple $(36,8,3,37)$ cannot be generated by Mordell's formulas since c must be the smaller of the odd numbers and

$$\begin{aligned} 3&=p^2-(n^2+m^2)\\ 37&=p^2+(n^2+m^2)\\ \hline 40&=2p^2\\ 20&=p^2; \qquad \text{so } p \text{ is not an integer.} \end{aligned}$$

This quadruple, however, can be generated from Carmichael's formulas [2], using $(m,n,p,q)=(1,4,2,4)$, that is,

$$\begin{aligned} a&=2mp+2nq \qquad & c&=p^2+q^2-(n^2+m^2),\\ b&=2np-2mq \qquad & d&=p^2+q^2+(n^2+m^2). \end{aligned} \tag{3}$$

By using an additional parameter, the Carmichael formulas generate a wider set of solutions where $m+n+p+q\equiv 1 \pmod 2$ and $(m,n,p,q)=1$.

In the formulas above, either three or four variables are needed to generate four other integers (a,b,c,d). In this paper, we present 2-parameter Pythagorean quadruple formulas where the two integral parameters are also part of the solution set. We shall call them **self-generating formulas**. These formulas will then be generalized to give **Pythagorean N-tuples** when a set of $(n-2)$ integers is given.

2. THE SELF-GENERATING QUADRUPLE FORMULAS

We use a and b to designate the two integer parameters that will generate the Pythagorean quadruples. The following theorem deals with the three possible cases arising from parity conditions imposed upon a and b.

Theorem 1: For positive integers a and b, where a or b or both are even, there exist integers c and d such that $a^2+b^2+c^2=d^2$. When a and b are both odd, no such integers c and d exist.

Case 1. If a and b are of opposite parity, then

$$c=(a^2+b^2-1)/2 \quad \text{and} \quad d=(a^2+b^2+1)/2. \tag{4}$$

Proof:

$$\begin{aligned} d^2-c^2 &= (d+c)(d-c) \\ &= \left[\left(\frac{a^2+b^2+1}{2}\right)+\left(\frac{a^2+b^2-1}{2}\right)\right]\left[\left(\frac{a^2+b^2+1}{2}\right)-\left(\frac{a^2+b^2-1}{2}\right)\right] \\ &= \left[\frac{2(a^2+b^2)}{2}\right]\left[\frac{2}{2}\right] \\ &= a^2+b^2. \end{aligned}$$

Therefore, $d^2=a^2+b^2+c^2$.

Since a and b differ in parity, c and d in (4) are integers.

Corollary: From (4), we see that c and d are consecutive integers. Therefore, $(a,b,c,d)=1$, even when $(a,b)\neq 1$.

Case 2. If a and b are both even, then

$$c=\left(\frac{a^2+b^2}{4}\right)-1 \quad \text{and} \quad d=\left(\frac{a^2+b^2}{4}\right)+1. \tag{5}$$

Proof:

$$\begin{aligned} 16(d^2-c^2) &= (a^2+b^2+4)^2-(a^2+b^2-4)^2 \\ &= 16(a^2+b^2). \end{aligned}$$

Therefore, $d^2=a^2+b^2+c^2$.

Since a and b are both even, c and d in (5) are integers.

Corollary: If $a-b\equiv 0 \pmod 4$, $(a^2+b^2)/4$ is even and c and d are consecutive odd integers, so $(a,b,c,d)=1$. But, if $a-b\equiv 2 \pmod 4$, $(a^2+b^2)/4$ is odd, c and d are consecutive even integers, and $(a,b,c,d)\neq 1$.

Case 3. If a and b are both odd, then $a^2\equiv b^2\equiv 1 \pmod 4$.

Since $c^2\equiv 0 \pmod 4$ or $c^2\equiv 1 \pmod 4$, and similarly for d^2, we have:

$$a^2+b^2+c^2\equiv 2 \pmod 4 \not\equiv d^2 \text{ for any integer } d;$$

or

$$a^2+b^2+c^2\equiv 3 \pmod 4 \not\equiv d^2 \text{ for any integer } d.$$

Hence, no Pythagorean quadruple exists in this case.

3. SELF-GENERATING PYTHAGOREAN N-TUPLES

The ideas and methods of proof for the self-generating quadruples can be generalized to the N-tuple case. We need to find formulas for generating integer N-tuples $(a_1, a_2, \ldots, a_n)$ when given a set of integer values for the $(n-2)$ members of the "parameter set" $S = (a_1, a_2, \ldots, a_{n-2})$. Analogous to the parity conditions imposed on the self-generating quadruple formulas, we introduce the variable T. Proofs of the formulas are left to the reader; they are similar to those for Theorem 1.

Theorem 2: Let $S = (a_1, a_2, \ldots, a_{n-2})$, where a_1 is an integer, and let $T =$ "# of odd integers in S."

If $T \not\equiv 2 \pmod 4$, then there exist integers a_{n-1} and a_n such that

$$a_1^2 + a_2^2 + \cdots + a_{n-1}^2 = a_n^2. \tag{6}$$

<u>**Case 1.**</u> Let $T \equiv 1 \pmod 2$, which implies that $T \equiv 1 \pmod 4$ or $T \equiv 3 \pmod 4$. Then, setting

$$a_{n-1} = [a_1^2 + a_2^2 + \cdots + a_{n-2}^2 - 1]/2,$$

and (7)

$$a_n = [a_1^2 + a_2^2 + \cdots + a_{n-2}^2 + 1]/2,$$

we have

$$\begin{aligned} a_n^2 - a_{n-1}^2 &= (a_n + a_{n-1})(a_n - a_{n-1}) \\ &= [2(a_1^2 + a_2^2 + \cdots + a_{n-2}^2)/2][2/2] \\ &= a_1^2 + a_2^2 + \cdots + a_{n-2}^2 \end{aligned}$$

as required.

<u>**Case 2.**</u> Let $T \equiv 0 \pmod 4$. Then, setting

$$a_{n-1} = [a_1^2 + a_2^2 + \cdots + a_{n-2}^2]/4 - 1$$

and (8)

$$a_n = [a_1^2 + a_2^2 + \cdots + a_{n-2}^2]/4 + 1,$$

we have

$$\begin{aligned} a_n^2 - a_{n-1}^2 &= (a_n + a_{n-1})(a_n - a_{n-1}) \\ &= [2(a_1^2 + a_2^2 + \cdots + a_{n-2}^2)/4][2] \\ &= a_1^2 + a_2^2 + \cdots + a_{n-2}^2 \end{aligned}$$

as required.

<u>**Case 3.**</u> Suppose $T \equiv 2 \pmod 4$. Then $a_1^2 + a_2^2 + \cdots + a_{n-2}^2 \equiv 2 \pmod 4$ And since either $a_{n-1}^2 \equiv 0 \pmod 4$ or $a_{n-1}^2 \equiv 1 \pmod 4$, we have

$$a_1^2 + a_2^2 + \cdots + a_{n-2}^2 + a_{n-1}^2 \equiv 2 \pmod 4 \not\equiv a_n^2 \text{ for any integer } a_n,$$

or

$$a_1^2 + a_2^2 + \cdots + a_{n-2}^2 + a_{n-1}^2 \equiv 3 \pmod 4 \not\equiv a_n^2 \text{ for any integer } a_n.$$

Hence, no Pythagorean quadruple N-tuple exists in this case.

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