

SELF-GENERATING PYTHAGOREAN QUADRUPLES AND N-TUPLES

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(Submitted December 1993)

1. INTRODUCTION

In a rectangular solid, the length of an interior diagonal is determined by the formula

$$a^2 + b^2 + c^2 = d^2, \quad (1)$$

where a, b, c are the dimensions of the solid and d is the diagonal.

When a, b, c , and d are integral, a **Pythagorean quadruple** is formed.

Mordell [1] developed a solution to this Diophantine equation using integer parameters (m, n , and p), where $m+n+p \equiv 1 \pmod{2}$ and $(m, n, p) = 1$. The formulas are:

$$\begin{aligned} a &= 2mp & c &= p^2 - (n^2 + m^2), \\ b &= 2np & d &= p^2 + (n^2 + m^2). \end{aligned} \quad (2)$$

However, the Pythagorean quadruple (36, 8, 3, 37) cannot be generated by Mordell's formulas since c must be the smaller of the odd numbers and

$$\begin{aligned} 3 &= p^2 - (n^2 + m^2) \\ 37 &= p^2 + (n^2 + m^2) \\ 40 &= 2p^2 \\ 20 &= p^2; \quad \text{so } p \text{ is not an integer.} \end{aligned}$$

This quadruple, however, can be generated from Carmichael's formulas [2], using $(m, n, p, q) = (1, 4, 2, 4)$, that is,

$$\begin{aligned} a &= 2mp + 2nq & c &= p^2 + q^2 - (n^2 + m^2), \\ b &= 2np - 2mq & d &= p^2 + q^2 + (n^2 + m^2). \end{aligned} \quad (3)$$

By using an additional parameter, the Carmichael formulas generate a wider set of solutions where $m+n+p+q \equiv 1 \pmod{2}$ and $(m, n, p, q) = 1$.

In the formulas above, either three or four variables are needed to generate four other integers (a, b, c, d). In this paper, we present 2-parameter Pythagorean quadruple formulas where the two integral parameters are also part of the solution set. We shall call them **self-generating formulas**. These formulas will then be generalized to give **Pythagorean N-tuples** when a set of $(n-2)$ integers is given.

2. THE SELF-GENERATING QUADRUPLE FORMULAS

We use a and b to designate the two integer parameters that will generate the Pythagorean quadruples. The following theorem deals with the three possible cases arising from parity conditions imposed upon a and b .

Theorem 1: For positive integers a and b , where a or b or both are even, there exist integers c and d such that $a^2 + b^2 + c^2 = d^2$. When a and b are both odd, no such integers c and d exist.

Case 1. If a and b are of opposite parity, then

$$c = (a^2 + b^2 - 1)/2 \quad \text{and} \quad d = (a^2 + b^2 + 1)/2. \quad (4)$$

Proof:

$$\begin{aligned} d^2 - c^2 &= (d+c)(d-c) \\ &= \left[\left(\frac{a^2 + b^2 + 1}{2} \right) + \left(\frac{a^2 + b^2 - 1}{2} \right) \right] \left[\left(\frac{a^2 + b^2 + 1}{2} \right) - \left(\frac{a^2 + b^2 - 1}{2} \right) \right] \\ &= \left[\frac{2(a^2 + b^2)}{2} \right] \left[\frac{2}{2} \right] \\ &= a^2 + b^2. \end{aligned}$$

Therefore, $d^2 = a^2 + b^2 + c^2$.

Since a and b differ in parity, c and d in (4) are integers.

Corollary: From (4), we see that c and d are consecutive integers. Therefore, $(a, b, c, d) = 1$, even when $(a, b) \neq 1$.

Case 2. If a and b are both even, then

$$c = \left(\frac{a^2 + b^2}{4} \right) - 1 \quad \text{and} \quad d = \left(\frac{a^2 + b^2}{4} \right) + 1. \quad (5)$$

Proof:

$$\begin{aligned} 16(d^2 - c^2) &= (a^2 + b^2 + 4)^2 - (a^2 + b^2 - 4)^2 \\ &= 16(a^2 + b^2). \end{aligned}$$

Therefore, $d^2 = a^2 + b^2 + c^2$.

Since a and b are both even, c and d in (5) are integers.

Corollary: If $a - b \equiv 0 \pmod{4}$, $(a^2 + b^2)/4$ is even and c and d are consecutive odd integers, so $(a, b, c, d) = 1$. But, if $a - b \equiv 2 \pmod{4}$, $(a^2 + b^2)/4$ is odd, c and d are consecutive even integers, and $(a, b, c, d) \neq 1$.

Case 3. If a and b are both odd, then $a^2 \equiv b^2 \equiv 1 \pmod{4}$.

Since $c^2 \equiv 0 \pmod{4}$ or $c^2 \equiv 1 \pmod{4}$, and similarly for d^2 , we have:

$$a^2 + b^2 + c^2 \equiv 2 \pmod{4} \neq d^2 \text{ for any integer } d,$$

or

$$a^2 + b^2 + c^2 \equiv 3 \pmod{4} \neq d^2 \text{ for any integer } d.$$

Hence, no Pythagorean quadruple exists in this case.

3. SELF-GENERATING PYTHAGOREAN N -TUPLES

The ideas and methods of proof for the self-generating quadruples can be generalized to the N -tuple case. We need to find formulas for generating integer N -tuples (a_1, a_2, \dots, a_n) when given a set of integer values for the $(n-2)$ members of the "parameter set" $S = (a_1, a_2, \dots, a_{n-2})$. Analogous to the parity conditions imposed on the self-generating quadruple formulas, we introduce the variable T . Proofs of the formulas are left to the reader; they are similar to those for Theorem 1.

Theorem 2: Let $S = (a_1, a_2, \dots, a_{n-2})$, where a_1 is an integer, and let $T = \text{"# of odd integers in } S\text{"}$.

If $T \not\equiv 2 \pmod{4}$, then there exist integers a_{n-1} and a_n such that

$$a_1^2 + a_2^2 + \dots + a_{n-2}^2 = a_n^2. \quad (6)$$

Case 1. Let $T \equiv 1 \pmod{2}$, which implies that $T \equiv 1 \pmod{4}$ or $T \equiv 3 \pmod{4}$. Then, setting

$$a_{n-1} = [a_1^2 + a_2^2 + \dots + a_{n-2}^2 - 1] / 2, \quad (7)$$

and

$$a_n = [a_1^2 + a_2^2 + \dots + a_{n-2}^2 + 1] / 2,$$

we have

$$\begin{aligned} a_n^2 - a_{n-1}^2 &= (a_n + a_{n-1})(a_n - a_{n-1}) \\ &= [2(a_1^2 + a_2^2 + \dots + a_{n-2}^2) / 2][2 / 2] \\ &= a_1^2 + a_2^2 + \dots + a_{n-2}^2 \end{aligned}$$

as required.

Case 2. Let $T \equiv 0 \pmod{4}$. Then, setting

$$a_{n-1} = [a_1^2 + a_2^2 + \dots + a_{n-2}^2] / 4 - 1 \quad (8)$$

and

$$a_n = [a_1^2 + a_2^2 + \dots + a_{n-2}^2] / 4 + 1,$$

we have

$$\begin{aligned} a_n^2 - a_{n-1}^2 &= (a_n + a_{n-1})(a_n - a_{n-1}) \\ &= [2(a_1^2 + a_2^2 + \dots + a_{n-2}^2) / 4][2] \\ &= a_1^2 + a_2^2 + \dots + a_{n-2}^2 \end{aligned}$$

as required.

Case 3. Suppose $T \equiv 2 \pmod{4}$. Then $a_1^2 + a_2^2 + \dots + a_{n-2}^2 \equiv 2 \pmod{4}$. And since either $a_{n-1}^2 \equiv 0 \pmod{4}$ or $a_{n-1}^2 \equiv 1 \pmod{4}$, we have

$$a_1^2 + a_2^2 + \dots + a_{n-2}^2 + a_{n-1}^2 \equiv 2 \pmod{4} \neq a_n^2 \text{ for any integer } a_n,$$

or

$$a_1^2 + a_2^2 + \dots + a_{n-2}^2 + a_{n-1}^2 \equiv 3 \pmod{4} \neq a_n^2 \text{ for any integer } a_n.$$

Hence, no Pythagorean quadruple N -tuple exists in this case.

ACKNOWLEDGMENTS

The author wishes to thank Dr. Ernest Eckert (University of South Carolina), Dr. Chuck Lanski (University of Southern California), Bill Graynom-Daly (Jefferson High School), and the referees for their valuable comments.

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1. L. J. Mordell. *Diophantine Equations*. London: Academic Press, 1969.
2. R. D. Carmichael. *Diophantine Analysis*. New York: John Wiley & Sons, 1915.

AMS Classification Number: 11D09



**FIBONACCI ENTRY POINTS AND PERIODS FOR PRIMES
100,003 THROUGH 415,993**

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