ON THE FIBONACCI NUMBERS WHOSE SUBSCRIPT IS A POWER

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1. AIM OF THE PAPER

The well-known identity (e.g., see [1], p. 127)

$$F_{2^n} = \prod_{i=1}^{n-1} L_{2^i} \tag{1.1}$$

led us to investigate the Fibonacci numbers of the form F_{k^n} with k and n positive integers. The principal aim of this note is to present some new identities involving F_{k^n}, some of which generalize (1.1). This is done in Sections 2 and 3. In Section 4, a first-order recurrence relation for F_{k^n} is established which involves certain combinatorial quantities whose properties will be investigated in a future paper. A glimpse of analogous results concerning the Lucas numbers L_{k^n} is caught in Section 5.

The formulas established in this note encompass the trivial case $n = 1$ under the usual assumptions

$$\prod_{i=a}^{b} f(i) = 1 \text{ if } b < a \tag{1.2}$$

and

$$\sum_{i=a}^{b} f(i) = 0 \text{ if } b < a. \tag{1.3}$$

2. MAIN RESULT

Proposition 1: If k is even and $n \geq 1$, then

$$F_{k^n} = F_k \prod_{i=1}^{n-1} \left[\sum_{j=1}^{k/2} L_{(2j-1)k^i} \right]. \tag{2.1}$$

We can immediately observe that, if $k = 2$, identity (2.1) reduces to (1.1).

Proof of Proposition 1: Write

$$F_{k^n} = F_k \cdot \frac{F_{k^2}}{F_k} \cdot \frac{F_{k^3}}{F_{k^2}} \cdot \ldots \cdot \frac{F_{k^n}}{F_{k^{n-1}}} = F_k \prod_{i=1}^{n-1} \frac{F_{k^{i+1}}}{F_{k^i}} \tag{2.2}$$

whence, following the notation used in [2] [namely, $R_s(t) = F_{st} / F_t$], we can rewrite (2.2) as

$$F_{k^n} = F_k \prod_{i=1}^{n-1} R_k(k^i). \tag{2.3}$$

Using (2.3) along with (2.1) of [2] yields

$$F_{k^n} = F_k \prod_{i=1}^{n-1} \left\{ \sum_{j=1}^{k/2} [F_{k^i(2j-1)-1} + F_{k^i(2j-1)+1}] \right\}. \tag{2.4}$$

The right-hand side of (2.4) clearly equals that of (2.1). Q.E.D.

Proposition 2: If k is odd and $n \geq 1$, then

$$F_{k^n} = (-1)^{(n-1)(k-1)/2} F_k \prod_{i=1}^{n-1} \left[1 + \sum_{j=1}^{(k-1)/2} (-1)^j L_{2jk^i} \right]. \tag{2.5}$$

Observe that, if $k = 3$, identity (2.5) reduces beautifully to

$$F_{3^n} = 2\prod_{i=1}^{n-1} (L_{2\cdot 3^i} - 1). \tag{2.6}$$

In order to prove Proposition 2, we need the identity

$$\sum_{i=1}^{r} (-1)^i L_{ai} = \frac{(-1)^{a+r} L_{ar} + (-1)^r L_{a(r+1)} - L_a - 2(-1)^a}{L_a + 1 + (-1)^a}, \tag{2.7}$$

which can readily be proved by using the Binet form for Lucas numbers and the geometric series formula. We also need the identity

$$L_{a+b} - (-1)^b L_{a-b} = 5F_a F_b, \tag{2.8}$$

which can be obtained from identities $I_{21} - I_{24}$ of [3].

Proof of Proposition 2: The proof has to be split into two cases according to the residue of k modulo 4.

Case 1. $k \equiv 1 \pmod 4$ [i.e., $(k-1)/2$ is even]
By (2.2), we can write

$$\begin{aligned}
F_{k^n} &= F_k \prod_{i=1}^{n-1} \frac{F_{k^{i+1}}}{F_{k^i}} = F_k \prod_{i=1}^{n-1} \left[1 + \frac{5F_{k^{i+1}} F_{k^i}}{5F_{k^i}^2} - 1 \right] \\
&= F_k \prod_{i=1}^{n-1} \left[1 + \frac{L_{k^{i+1}-k^i} + L_{k^{i+1}+k^i}}{L_{2k^i} + 2} - 1 \right] \quad \text{(by (2.8) and } I_{17} \text{ of [3])} \\
&= F_k \prod_{i=1}^{n-1} \left[1 + \frac{L_{2k^i(k-1)/2} + L_{2k^i(k+1)/2} - L_{2k^i} - 2}{L_{2k^i} + 2} \right] \\
&= F_k \prod_{i=1}^{n-1} \left[1 + \sum_{h=1}^{(k-1)/2} (-1)^h L_{2hk^i} \right] \quad \text{[by (2.7)]}.
\end{aligned}$$

Observe that, since $(k-1)/2$ is even by hypothesis, the above expression does not vary if we multiply it by $(-1)^{(n-1)(k-1)/2}$.

Case 2. $k \equiv 3 \pmod 4$ [i.e., $(k-1)/2$ is odd]
Analogously, we can write

$$F_{k^n} = (-1)^{n-1} F_k \prod_{i=1}^{n-1} \left[-\frac{5F_{k^{i+1}} F_{k^i}}{5F_{k^i}^2} - 1 + 1 \right]$$

$$= (-1)^{n-1} F_k \prod_{i=1}^{n-1} \left[1 + \frac{-L_{2k^i(k-1)/2} - L_{2k^i(k+1)/2} - L_{2k^i} - 2}{L_{2k^i} + 2} \right]$$

$$= (-1)^{n-1} F_k \prod_{i=1}^{n-1} \left[1 + \sum_{h=1}^{(k-1)/2} (-1)^h L_{2hk^i} \right].$$

Observe that, since $(k-1)/2$ is odd by hypothesis, the factor $(-1)^{n-1}$ in the above expression can be rewritten as $(-1)^{(n-1)(k-1)/2}$. Q.E.D.

3. RELATED RESULTS

Some results related to those established in Section 2 can be obtained readily. Observe that, if the exponent n in (2.1) is composite (say, $n = st$), then F_{k^n} (k even) can be expressed as

$$F_{k^n} = F_{k^{st}} = F_{k^s} \prod_{i=1}^{t-1} \left[\sum_{j=1}^{k^s/2} L_{(2j-1)k^{si}} \right] \quad (k \text{ even}), \tag{3.1}$$

where s and t can obviously be interchanged. For example, by (1.1) and (3.1), we have

$$F_{2^{2n}} = F_{4^n} = \prod_{i=1}^{2n-1} L_{2^i} = F_{2^n} \sum_{j=1}^{2^{n-1}} L_{(2j-1)2^n} = 3 \prod_{i=1}^{n-1} (L_{4^i} + L_{3 \cdot 4^i}). \tag{3.2}$$

For k odd, the analog of (3.1) can be obtained immediately.

An expression analogous to (2.1) can be established for F_{mk^n} when mk is even. If mk is odd, the corresponding expression is somewhat unattractive and its presentation is omitted.

Proposition 3: If $n \geq 1$,

$$F_{mk^n} = F_{mk} \prod_{i=1}^{n-1} \left[\sum_{j=1}^{k/2} L_{(2j-1)mk^i} \right] \quad (k \text{ even}, m \text{ arbitrary}) \tag{3.3}$$

and

$$F_{mk^n} = F_{mk} \prod_{i=1}^{n-1} \left[1 + \sum_{j=1}^{(k-1)/2} L_{2jmk^i} \right] \quad (k \text{ odd}, m \text{ even}). \tag{3.4}$$

Proof: Write

$$F_{mk^n} = F_{mk} \prod_{i=1}^{n-1} \frac{F_{mk^{i+1}}}{F_{mk^i}} = F_{mk} \prod_{i=1}^{n-1} R_k(mk^i)$$

and use (2.1) and (2.2) of [2]. Q.E.D.

If we let $m = k$ in (3.3), we see that an equivalent form for (2.1) is

$$F_{k^n} = F_{k^2} \prod_{i=1}^{n-2} \left[\sum_{j=1}^{k/2} L_{(2j-1)k^{i+1}} \right] \quad (k \text{ even}, \; n \geq 2). \tag{3.5}$$

Moreover, if we let $n = u + v$ $(u, v \geq 1)$ and $m = k^u$ in (3.3), we get the relation

$$F_{k^n} = F_{k^{u+v}} = F_{k^u k^v} = F_{k^{u+1}} \prod_{i=1}^{v-1} \left[\sum_{j=1}^{k/2} L_{(2j-1)k^{u+i}} \right] \quad (k \text{ even}), \tag{3.6}$$

which generalizes (3.5).

4. A RECURRENCE RELATION FOR F_{k^n}

A problem [6] that appeared in this journal led us to discover the first-order nonlinear homogeneous recurrence relation

$$F_{3^{n+1}} = 5F_{3^n}^3 - 3F_{3^n} \quad (n \geq 0). \tag{4.1}$$

The aim of this section is to obtain an analogous relation valid for all positive subscripts k^{n+1} with k *odd* and n an arbitrary nonnegative integer.

Proposition 4: If k is a positive odd integer and n is a nonnegative integer, then

$$F_{k^{n+1}} = 5^{(k-1)/2} F_{k^n}^k - \sum_{i=0}^{(k-3)/2} 5^i C_{i,k} F_{k^n}^{2i+1}, \tag{4.2}$$

where the coefficients $C_{i,k}$ are given by

$$C_{i,k} = (-1)^{(k+1)/2+i} \binom{(k+1)/2+i}{2i+1} \frac{k}{(k+1)/2+i} \quad [0 \leq i \leq (k-3)/2]. \tag{4.3}$$

As an example, for $k = 3, 5, 7$, and 9, (4.2) gives (4.1),

$$F_{5^{n+1}} = 25F_{5^n}^5 - 25F_{5^n}^3 + 5F_{5^n}, \tag{4.4}$$

$$F_{7^{n+1}} = 125F_{7^n}^7 - 175F_{7^n}^5 + 70F_{7^n}^3 - 7F_{7^n}, \tag{4.5}$$

and

$$F_{9^{n+1}} = 625F_{9^n}^9 - 1125F_{9^n}^7 + 675F_{9^n}^5 - 150F_{9^n}^3 + 9F_{9^n}, \tag{4.6}$$

respectively.

Proof of Proposition 4: First, let us write

$$F_{k^n}^k = \frac{1}{(\sqrt{5})^k} \sum_{j=0}^{k} \binom{k}{j} (-1)^j \beta^{jk^n} \alpha^{(k-j)k^n}, \tag{4.7}$$

where $\alpha = 1 - \beta = (1 + \sqrt{5})/2$. After several simple but tedious manipulations involving the use of the Binet forms for Fibonacci and Lucas numbers, and the relation $\alpha\beta = -1$, (4.7) yields the following identities which may be of some interest *per se*:

$$F_{k^n}^k = \frac{1}{5^{(k-1)/2}}\left[F_{k^{n+1}} + \sum_{j=1}^{(k-1)/2} \binom{k}{j} F_{(k-2j)k^n}\right] \quad (k \text{ odd}, \ n \geq 0), \tag{4.8}$$

$$F_{k^n}^k = \frac{1}{5^{k/2}}\left[(-1)^{k/2}\binom{k}{k/2} + \sum_{j=0}^{(k-2)/2} (-1)^j \binom{k}{j} L_{(k-2j)k^n}\right] \quad (k \text{ even}, \ n \geq 1). \tag{4.9}$$

By (4.8), we immediately obtain

$$F_{k^{n+1}} = 5^{(k-1)/2} F_{k^n}^k - \sum_{j=1}^{(k-1)/2} \binom{k}{j} F_{(k-2j)k^n} \quad (k \text{ odd}). \tag{4.10}$$

Then, using (4.10) along with Theorem 1 of [4] leads to the expression

$$F_{k^{n+1}} = 5^{(k-1)/2} F_{k^n}^k - \sum_{j=1}^{(k-1)/2} \sum_{i=0}^{(k-1)/2-j} (-1)^{(k-1)/2+i-j} \frac{k-2j}{(k+1)/2+i-j} 5^i \binom{k}{j} \binom{(k+1)/2+i-j}{2i+1} F_{k^n}^{2i+1} \tag{4.11}$$

which, after reversing the summation order, can be rewritten as

$$F_{k^{n+1}} = 5^{(k-1)/2} F_{k^n}^k - \sum_{i=0}^{(k-1)/2} 5^i A_{i,k} F_{k^n}^{2i+1}, \tag{4.12}$$

where

$$A_{i,k} = (-1)^i \sum_{j=1}^{(k-1)/2-i} (-1)^{(k-1)/2-j} \frac{k-2j}{(k+1)/2+i-j} \binom{k}{j} \binom{(k+1)/2+i-j}{2i+1}. \tag{4.13}$$

Since $A_{(k-1)/2,k} = 0$ by (1.3), expression (4.12) becomes

$$F_{k^{n+1}} = 5^{(k-1)/2} F_{k^n}^k - \sum_{i=0}^{(k-3)/2} 5^i A_{i,k} F_{k^n}^{2i+1}. \tag{4.14}$$

Now it remains to show that the numbers $A_{i,k}$ [defined by (4.13)] and the numbers $C_{i,k}$ [defined by (4.3)] coincide. To do this, consider the combinatorial identity

$$\sum_{j=1}^{m} (-1)^j \frac{k-2j}{k-m-j} \binom{k}{j} \binom{k-m-j}{m-j} = -\frac{k}{k-m} \binom{k-m}{m} \quad [1 \leq m \leq (k-1)/2], \tag{4.15}$$

which can be obtained by [5, p. 58], and replace m by $(k-1)/2-i$ in (4.15) to obtain the desired result $C_{i,k} = A_{i,k}$. Q.E.D.

5. CONCLUDING REMARKS

Some properties of the numbers F_{k^n} have been investigated in this note. In particular, expressions for these numbers in terms of products involving Lucas numbers have been established. Analogous expressions for L_{k^n} appeared to be rather unpleasant, so we confine ourselves to show some partial results whose proofs are left to the perseverance of the reader. In particular, we show the identity

$$L_{k^n} = 2 + (L_{k^2} - 2) \prod_{i=1}^{n-2} \left[\sum_{j=1}^{k/2} L_{(2j-1)k^{i+1}/2} \right]^2 \quad (k \text{ even}, \ n \geq 2) \tag{5.1}$$

which, for $k = 2$, reduces to

$$L_{2^n} = 2 + 5 \prod_{i=1}^{n-2} L_{2^i}^2 \quad (n \geq 2). \tag{5.2}$$

We also have

$$L_{3^n} = 4 \prod_{i=1}^{n-1} (L_{2\cdot 3^i} + 1) \quad (n \geq 1). \tag{5.3}$$

Observe that the identity

$$F_{2\cdot 3^n} = 8 \prod_{i=1}^{n-1} (L_{2\cdot 3^i}^2 - 1) = 8 \prod_{i=1}^{n-1} (L_{4\cdot 3^i} + 1) \quad (n \geq 1) \tag{5.4}$$

can be obtained either by (2.6) and (5.3), or by (3.4).

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