GCD AND LCM POWER MATRICES

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1. INTRODUCTION

Let $S=\{x_1, x_2, \ldots, x_n\}$ be a set of distinct positive integers. By (x_i, x_j) and $[x_i, x_j]$, we denote the greatest common divisor (GCD) and the least common multiple (LCM) of x_i and x_j, respectively.

The matrix (S) (resp. $[S]$) having (x_i, x_j) (resp. $[x_i, x_j]$) as its i, j-entry is called the GCD (resp. LCM) matrix defined on S.

A set is called factor-closed if it contains every divisor of each of its members. A set S is gcd-closed if $(x_i, x_j) \in S$ for any i and j $(1 \le i, j \le n)$.

Smith [6] and Beslin and Ligh [3] discussed (S) and $\det(S)$, the determinant of (S). They proved that $\det(S) = \phi(x_1) \ldots \phi(x_n)$, where ϕ is Euler's totient, if S is factor-closed. Beslin and Ligh [4] gave a formula for $\det(S)$ when S is gcd-closed.

Smith [6] and Beslin [2] considered the LCM matrix $[S]$ when S is factor-closed. In 1992, Boueque and Ligh [1] gave a formula for $\det[S]$ when S is gcd-closed. They also obtained formulas for $(S)^{-1}$ and $[S]^{-1}$, the inverses of (S) and $[S]$.

Let r be a real number. The matrix $(S^r) = (a_{ij})$, where $a_{ij} = (x_i, x_j)^r$, is called the GCD power matrix defined on S; the matrix $[S^r] = (b_{ij})$, where $b_{ij} = [x_i, x_j]^r$, is called the LCM power matrix defined on S.

In this paper the results mentioned above are generalized by giving formulas for (S^r), $[S^r]$, $\det(S^r)$, and $\det[S^r]$ on factor-closed sets and gcd-closed sets, respectively. Making use of the Möbius matrix, which is a generalization of the Möbius function μ, we shall give the inverse matrices of (S^r) and $[S^r]$.

All known results about (S) and $[S]$ are just the particular cases of the theory of (S^r) and $[S^r]$ on condition that $r=1$.

One of the problems raised by Beslin [2] are solved. Some conjectures are put forward.

2. JORDAN'S TOTIENT

For any positive integer n and real r, we define

$$J_r(n) = n^r \prod_{p|n} \left(1 - \frac{1}{p^r}\right).$$

The function J_r is usually called *Jordan's totient.*

Theorem 1: If $n \ge 1$ and r is real, then

$$\sum_{d|n} J_r(d) = n^r. \tag{2.1}$$

Proof: By the definition of J_r, when $n = p_1^{a_1} \dots p_k^{a_k}$,

$$J_r(n) = n^r\left(1-\frac{1}{p_1^r}\right)\dots\left(1-\frac{1}{p_k^r}\right) = n^r\sum_{d|n}\frac{\mu(d)}{d^r} = \sum_{d|n}\mu(d)\left(\frac{n}{d}\right)^r. \tag{2.2}$$

Equation (2.2) and the Möbius inversion formula give (2.1). □

3. MÖBIUS MATRICES

Let $S = \{x_1, \dots, x_n\}$ be ordered by $x_1 < x_2 < \dots < x_n$. We define $U = (u_{ij})$, where

$$u_{ij} = \begin{cases} 1 & \text{if } x_i | x_j, \\ 0 & \text{otherwise.} \end{cases} \tag{3.1}$$

Our purpose is to find $M = (\mu_{ij}) = U^{-1}$. As S is ordered, U is an upper triangular matrix. It is well known that the inverse of an upper triangular matrix is also an upper triangular matrix. Hence,

$$\mu_{ij} = \mu(x_i, x_j) = 0, \text{ if } i > j \text{ (i.e., } x_i > x_j). \tag{3.2}$$

Since $M = U^{-1}$, we have $\sum_{k=1}^n u_{ik}u_{kj} = \delta_{ij}$. Using (3.1),

$$\sum_{x_k | x_j}^{n} \mu(x_i, x_k) = \delta_{ij}. \tag{3.3}$$

When $i = j$, by (3.2) and (3.3), we have

$$\mu(x_i, x_i) = 1 \quad (i = 1, 2, \dots, n). \tag{3.4}$$

When $i < j$, by (3.3), we have

$$\mu_{ij} = \mu(x_i, x_j) = -\sum_{\substack{x_k | x_j \\ x_k < x_j}} \mu(x_i, x_k). \tag{3.5}$$

Theorem 2: Function $\mu(x, y)$ is multiplicative.

Proof: $\mu(x, y)$ may be written as $\mu(p_1^{a_1} \dots p_s^{a_s}, p_1^{b_1} \dots p_s^{b_s})$, where $a_i \geq 0$, $b_i \geq 0$, but $a_i + b_i > 0$, $i = 1, 2, \dots, s$. First, for any $a_i \geq 0$ $(i = 1, 2, \dots, s)$, by (3.2) and (3.4), we have

$$\mu(p_1^{a_1} \dots p_s^{a_s}, 1) = \mu(p_1^{a_1}, 1) \dots \mu(p_s^{a_s}, 1). \tag{3.6}$$

Next, we make an inductive hypothesis:

$$\mu(p_1^{a_1} \dots p_s^{a_s}, p_1^{i_1} \dots p_s^{i_s}) = \mu(p_1^{a_1}, p_1^{i_1}) \dots \mu(p_s^{a_s}, p_s^{i_s}), \tag{3.7}$$

for $(0, \dots, 0) \leq (i_1, \dots, i_s) < (b_1, \dots, b_s)$, which may be abbreviated $(0) \leq (i) < (b)$.

Note that $(i_1, \dots, i_s) = (b_1, \dots, b_s)$ means $i_k = b_k$, $k = 1, 2, \dots, s$; $(i_1, \dots, i_s) < (b_1, \dots, b_s)$ means $i_k \leq b_k$, and there exists at least a t such that $i_t < b_t$ $(1 \leq t \leq s)$.

When $(a) \neq (b)$, by (3.5) and (3.7), we have

$$\begin{aligned}
\mu(p_1^{a_1}\dots p_s^{a_s},\ p_1^{b_1}\dots p_s^{b_s}) &= -\sum_{(0)\le(i)<(b)} \mu(p_1^{a_1}\dots p_s^{a_s},\ p_1^{i_1}\dots p_s^{i_s})\\
&= -\sum_{(0)\le(i)<(b)} \mu(p_1^{a_1}, p_1^{i_1})\cdots(p_s^{a_s}, p_s^{i_s})\\
&= \left[\binom{s}{s}(-1)^s + \binom{s}{s-1}(-1)^{s-1} + \cdots + \binom{s}{1}(-1)^1\right]\mu(p_1^{a_1}, p_1^{b_1})\cdots(p_s^{a_s}, p_s^{b_s})\\
&= -[(1-1)^s - 1]\mu(p_1^{a_1}, p_1^{b_1})\cdots(p_s^{a_s}, p_s^{b_s})\\
&= \mu(p_1^{a_1}, p_1^{b_1})\cdots(p_s^{a_s}, p_s^{b_s}).
\end{aligned}$$

In summing, we consider all combinatorial possibilities of $0 \le i_k < b_k$ and $i_k = b_k$ satisfying $(0) \le (i) < (b)$; also,

$$\sum_{0\le i_k<b_k} \mu(p_k^{a_k}, p_k^{i_k}) = -\mu(p_k^{a_k}, p_k^{b_k})$$

has been used.

When $(a) = (b)$, by (3.4), we have

$$\mu(p_1^{a_1}\dots p_s^{a_s},\ p_1^{b_1}\dots p_s^{b_s}) = 1 = \mu(p_1^{a_1}, p_1^{b_1})\dots\mu(p_s^{a_s}, p_s^{b_s}). \ \square$$

Theorem 3: The generalized Möbius function

$$\mu(x, y) = \begin{cases} (-1)^s & \text{if } \frac{y}{x} = p_1\dots p_s,\ s>0,\\ 1 & \text{if } x = y,\\ 0 & \text{otherwise.}\end{cases}$$

Proof: Let p be a prime. By (3.2), (3.4), and (3.5),

$$\mu(p^m, p^n) = 0, \text{ if } m > n; \quad \mu(p^m, p^m) = 1;$$
$$\mu(p^m, p^{m+1}) = -\mu(p^m, p^m) = -1.$$

When $k \ge 2$, we have

$$\begin{aligned}
\mu(p^m, p^{m+k}) &= -\sum_{0\le i<k} \mu(p^m, p^{m+i})\\
&= -\sum_{0\le i<k-1} \mu(p^m, p^{m+i}) - \mu(p^m, p^{m+k-1})\\
&= \mu(p^m, p^{m+k-1}) - \mu(p^m, p^{m+k-1}) = 0.
\end{aligned}$$

These results and Theorem 2 complete the proof. $\square$

4. GCD POWER MATRICES ON FACTOR-CLOSED SETS

Let $S = \{x_1, x_2, \dots, x_n\}$ be an ordered set of distinct positive integers, and $\overline{S} = \{y_1, y_2, \dots, y_m\}$, which is ordered by $y_1 < y_2 < \cdots < y_m$, be a minimal factor-closed set containing S. We call $\overline{S}$ the factor-closed closure of S.

Theorem 4: Let $S = \{x_1, \dots, x_n\}$ be an ordered set of distinct positive integers, and $\overline{S} = \{y_1, \dots, y_m\}$ the factor-closed closure of S. Then the GCD power matrix on S, i.e.,

$$(S^r) = E^T G_r E, \tag{4.1}$$

where

$$G_r = \mathrm{diag}(J_r(y_1), \ldots, J_r(y_m)), \tag{4.2}$$

$$E = (e_{ij}), \quad e_{ij} = \begin{cases} 1 & \text{if } y_i \mid x_j, \\ 0 & \text{otherwise.} \end{cases} \tag{4.3}$$

Proof: By (2.1), we have

$$(E^t G_r E)_{ij} = \sum_{k=1}^{m} e_{ki} J_r(y_k) e_{kj} = \sum_{\substack{y_k \mid x_i \\ y_k \mid x_j}} J_r(y_k) = \sum_{y_k \mid (x_i, x_j)} J_r(y_k)$$

$$= \sum_{d \mid (x_i, x_j)} J_r(d) = (x_i, x_j)^r = (S^r)_{ij}. \quad \square$$

Theorem 5: Let S be factor-closed, then we have

$$\det(S^r) = J_r(x_1) \ldots J_r(x_n). \tag{4.4}$$

Proof: When S is factor-closed, $S = \overline{S}$, and the matrix E is equal to U, which is defined as (3.1), and is a triangular matrix with the diagonal $(1, 1, \ldots, 1)$. We have

$$\det(S^r) = (\det U)^2 \det G_r = \det G_r = J_r(x_1) \ldots J_r(x_n). \quad \square$$

When S is arbitrary, $\det(S^r)$ can be calculated by the Cauchy-Binet formula [8]. We omit this here for succinctness.

Remark 1: Letting $r = 1$ in (4.4), we obtain the well-known results of Smith [6] and of Beslin and Ligh [3]:

$$\det(S) = J_1(x_1) \ldots J_1(x_n) = \phi(x_1) \ldots \phi(x_n).$$

Remark 2: By (4.1), we have the reciprocal GCD power matrix

$$(S^{-r}) = E^T G_{-r} E. \tag{4.5}$$

Hence, if S is factor-closed, we have

$$\det(S^{-r}) = J_{-r}(x_1) \ldots J_{-r}(x_n), \tag{4.6}$$

$$\det(S^{-1}) = J_{-1}(x_1) \ldots J_{-1}(x_n). \tag{4.7}$$

In fact, (4.7) is exactly Corollary 1 of Beslin [2]. It is evident that the function $g(n)$ introduced by Beslin in [2] and by Bourque and Ligh in [1] is none other than Jordan's totient function $J_{-1}(n)$.

5. LCM POWER MATRICES ON FACTOR-CLOSED SETS

In this section, we shall turn our attention to the LCM power matrix.

Theorem 6: Let S and $\overline{S}$ be defined as in Theorem 4. Then we have the LCM power matrix

$$[S^r] = D_r E^T G_{-r} E D_r, \tag{5.1}$$

where

$$D_r = \operatorname{diag}(x_1^r, \ldots, x_n^r), \tag{5.2}$$

G_{-r} and E are defined by (4.2) and (4.3).

Proof: By (4.5), we have

$$\begin{aligned}(D_r E^T G_{-r} E D_r)_{ij} &= (D_r (S^{-r}) D_r)_{ij} = x_i^r (S^{-r})_{ij} x_j^r \\ &= \frac{x_i^r x_j^r}{(x_i, x_j)^r} = [x_i, x_j]^r = [S^r]_{ij}. \quad \square\end{aligned}$$

Theorem 7: If S is factor-closed, then the determinant

$$\begin{aligned}\det[S^r] &= x_1^{2r} \ldots x_n^{2r} J_{-r}(x_1) \ldots J_{-r}(x_n) \\ &= J_r(x_1) \ldots J_r(x_n) \pi_r(x_1) \ldots \pi_r(x_n),\end{aligned} \tag{5.3}$$

where π_r is multiplicative and for the prime power p^m, $\pi_r(p^m) = -p^r$.

Proof: By (5.1) and the fact that $E = U$, we have

$$\det[S^r] = \prod_{i=1}^{n} x_i^{2r} J_{-r}(s_i) \quad \text{and} \quad x_i^{2r} J_{-r}(x_i) = J_r(x_i) \pi_r(x_i).$$

This completes the proof. $\square$

Remark 3: Letting $r = 1$ in (5.3), we shall have Corollary 3 of Beslin [2] immediately.

On the basis of (4.4) and (5.3), we have

Theorem 8: If S is factor-closed, then

$$\frac{\det[S^r]}{\det(S^r)} = \prod_{i=1}^{n} \pi_r(x_i), \tag{5.4}$$

$$\frac{\det[S]}{\det(S)} = \prod_{i=1}^{n} \pi(x_i), \tag{5.5}$$

where $\pi(n)$ is multiplicative, and $\pi(p^k) = -p$, for the prime power p^k.

Remark 4: By (5.4) and (5.5), we know that $[S]$ and $[S^r]$ are not positive definite.

Remark 5: Let $\omega(x)$ denote the number of distinct prime factors of x, and $\Omega = \omega(x_1) + \cdots + \omega(x_n)$. By Theorem 8, we know that $\det[S]$ and $\det[S^r]$ are positive, if Ω is even; they are negative if Ω is odd, for factor-closed S. This solves the second of the problems put forward by Beslin in [2].

6. INVERSES OF (S^r) AND $[S^r]$ ON FACTOR-CLOSED SETS

In Section 3, we obtained $M = (\mu(x_1, x_j)) = U^{-1}$. Now we shall give $(S^r)^{-1}$ and $[S^r]^{-1}$, the inverses of (S^r) and $[S^r]$, respectively.

Theorem 9: Let S be factor-closed, then $(S^r)^{-1}=(a_{ij})$ and $[S^r]^{-1}=(b_{ij})$, where

$$a_{ij}=\sum_{[x_i,x_j]\|x_k}\frac{\mu(x_i,x_k)\mu(x_j,x_k)}{J_r(x_k)};\tag{6.1}$$

$$b_{ij}=\sum_{[x_i,x_j]\|x_k}\left(\frac{x_k}{x_i}\right)^r\left(\frac{x_k}{x_j}\right)^r\frac{\mu(x_i,x_k)\mu(x_j,x_k)}{J_r(x_k)\pi_r(x_k)}.\tag{6.2}$$

Proof: When S is factor-closed, we have $E=U$. By (4.1),

$$\begin{aligned}a_{ij}&=(U^{-1}G_r^{-1}(U^{-1})^T)_{ij}=(MG_r^{-1}M^T)_{ij}\\&=\sum_{k=1}^n\mu_{ik}(J_r(x_k))^{-1}\mu_{jk}=\sum_{[x_i,x_j]\|x_k}\frac{\mu(x_i,x_k)\mu(x_j,x_k)}{J_r(x_k)}.\end{aligned}$$

By (5.2), we have

$$\begin{aligned}b_{ij}&=(D_r^{-1}U^{-1}G_{-r}^{-1}(U^{-1})^TD_r^{-1})_{ij}=(D_r^{-1}MG_{-r}^{-1}M^TD_r^{-1})_{ij}\\&=\sum_{k=1}^n x_i^{-r}\mu_{ik}(J_{-r}(x_k))^{-1}\mu_{jk}x_j^{-r}=\frac{1}{x_i^rx_j^r}\sum_{\substack{x_i|x_k\\x_j|x_k}}\frac{\mu(x_i,x_k)\mu(x_j,x_k)}{J_{-r}(x_k)}\\&=\sum_{[x_i,x_j]\|x_k}\left(\frac{x_k}{x_i}\right)^r\left(\frac{x_k}{x_j}\right)^r\frac{\mu(x_i,x_k)\mu(x_j,x_k)}{J_r(x_k)\pi_r(x_k)}.\quad\square\end{aligned}$$

Remark 6: Theorem 9 is a generalization of Theorems 1 and 2 of Bourque and Ligh [1].

7. (S^r) AND $[S^r]$ ON GCD-CLOSED SETS

Let $\alpha_r(x_i)$, $i=1,2,\dots,n$, be defined by

$$\alpha_r(x_i)=\sum_{\substack{d|x_i\\d\nmid x_t\\x_t<x_i}}J_r(d).\tag{7.1}$$

Using the principle of cross-classification [7] and (2.1), we can prove

Theorem 10: Let $S=\{x_1,x_2,\dots,x_n\}$ be ordered by $x_1<x_2<\cdots<x_n$ and let $\alpha_r(x_i)$ be defined by (7.1). Then

$$\begin{aligned}\alpha_r(x_i)&=x_i^r-\sum_{1\le j<1}(x_j,x_i)^r+\sum_{1\le j<k<i}(x_j,x_k,x_i)^r-\cdots\\&\quad+(-1)^{i-1}(x_1,x_2,\dots,x_i)^r,\ i=1,\dots,n.\end{aligned}\tag{7.2}$$

Theorem 11: Let S be gcd-closed, then

$$(S^r)=U^TA_rU,\tag{7.3}$$

$$[S^r]=D_rU^TA_{-r}UD_r,\tag{7.4}$$

where $A_r=\operatorname{diag}(\alpha_r(x_1),\dots,\alpha_r(x_n))$, U and D_r are defined in (3.1) and (5.2), respectively.

Proof: The proof of (7.3) is simple. We shall prove only (7.4).

$$
\begin{aligned}
(D_r U^T A_{-r} U D_r)_{ij} &= \sum_{k=1}^{n} x_i^r u_{ki} \alpha_{-r}(x_k) u_{kj} x_j^r = x_i^r x_j^r \sum_{\substack{x_k \mid x_i \\ x_k \mid x_j}} \alpha_{-r}(x_k) \\
&= x_i^r x_j^r \sum_{x_k \mid (x_i, x_j)} \sum_{\substack{d \mid x_k \\ d \nmid x_t \\ x_t < x_k}} J_{-r}(d) = x_i^r x_j^r \sum_{d \mid (x_i, x_j)} J_{-r}(d) \\
&= x_i^r x_j^r / (x_i, x_j)^r = [x_i, x_j]^r = [S^r]_{ij}. \quad \square
\end{aligned}
$$

On the basis of Theorem 11, it is easy to prove

Theorem 12: Let S be gcd-closed, then

$$\det(S^r) = \prod_{i=1}^{n} \alpha_r(x_i), \tag{7.5}$$

$$\det[S^r] = \prod_{i=1}^{n} x_i^{2r} \alpha_{-r}(x_i). \tag{7.6}$$

Remark 7: Letting $r = 1$, equation (7.5) becomes Corollary 1 of Beslin and Ligh [4] and equation (7.6) becomes Theorem 5 of Bourque and Ligh [1].

8. INVERSES OF (S^r) AND $[S^r]$ ON GCD-CLOSED SETS

When S is gcd-closed, the inverse matrices $(S^r)^{-1}$ and $[S^r]^{-1}$ can be derived easily from Theorem 11. For future reference, we present the formulas without proof.

Theorem 13: Let S be gcd-closed, then

$$(S^r)^{-1} = (c_{ij}) \quad \text{and} \quad [S^r]^{-1} = (d_{ij}),$$

where

$$c_{ij} = \sum_{[x_i, x_j] \mid x_k} \frac{\mu(x_i, x_k)\mu(x_j, x_k)}{\alpha_r(x_k)}, \tag{8.1}$$

$$d_{ij} = \frac{1}{x_i^r x_j^r} \sum_{[x_i, x_j] \mid x_k} \frac{\mu(x_i, x_k)\mu(x_j, x_k)}{\alpha_{-r}(x_k)}. \tag{8.2}$$

Remark 8: We make the following conjectures, which are similar to the conjecture of Bourque and Ligh [1]:

1. If S is gcd-closed and $r \neq 0$, the LCM power matrix $[S^r]$ is invertible.

2. Let $S = \{x_1, x_2, \ldots, x_n\}$ be an ordered set of distinct positive integers and $r \neq 0$, then

$$\frac{1}{x_n^r} - \sum_{1 \le i < n} \frac{1}{(x_i, x_n)^r} + \sum_{1 \le i < j < n} \frac{1}{(x_1, x_j, x_n)^r} - \cdots + (-1)^{n-1} \frac{1}{(x_1, x_2, \ldots, x_n)^r} \neq 0.$$

3. Let $A = \{a_1, a_2, \ldots, a_n\}$ be a set of distinct positive integers and $a_i > 1$ $(i = 1, \ldots, n)$, $r \neq 0$, then

$$1 - \sum_{1 \le i \le n} a_i^r + \sum_{1 \le i < j \le n} [a_i, a_j]^r - \cdots + (-1)^n [a_1, \ldots, a_n]^r \neq 0.$$

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