ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
Stanley Rabinowitz

Please send all material for ELEMENTARY PROBLEMS AND SOLUTIONS to Dr. STANLEY RABINOWITZ; 12 VINE BROOK RD; WESTFORD, MA 01886-4212 USA. Correspondence may also be sent to the problem editor by electronic mail to Fibonacci@MathPro.com *on Internet. All correspondence will be acknowledged.*

Each solution should be on a separate sheet (or sheets) and must be received within six months of publication of the problem. Solutions typed in the format used below will be given preference. Proposers of problems should normally include solutions. Although this Elementary Problem section does not insist on original problems, we do ask that proposers inform us of the history of the problem, if it is not original. A problem should not be submitted elsewhere while it is under consideration for publication in this column.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \ F_0 = 0, \ F_1 = 1;$$
$$L_{n+2} = L_{n+1} + L_n, \ L_0 = 2, \ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-814 ***Proposed by M. N. Deshpande, Institute of Science, Nagpur, India***

Show that for each positive integer n, there exists a constant C_n such that $F_{2n+2i}F_{2i} - C_n$ and $F_{2n+2i+1}F_{2i+1} - C_n$ are both perfect squares for all positive integers i.

B-815 ***Proposed by Paul S. Bruckman, Highwood, IL***

Let $K(a,b,c) = a^3 + b^3 + c^3 - 3abc$. Show that if x_1, x_2, x_3, y_1, y_2, and y_3 are integers, then there exist integers z_1, z_2, and z_3 such that

$$K(x_1, x_2, x_3) \cdot K(y_1, y_2, y_3) = K(z_1, z_2, z_3).$$

B-816 ***Proposed by Mohammad K. Azarian, University of Evansville, Evansville, IN***

Let i, j, and k be any three positive integers. Show that

$$\frac{F_jF_k}{F_i + F_iF_jF_k} + \frac{F_kF_i}{F_j + F_iF_jF_k} + \frac{F_iF_j}{F_k + F_iF_jF_k} < 2.$$

B-817 ***Proposed by Kung-Wei Yang, Western Michigan University, Kalamazoo, MI***

Show that

$$\sqrt[k]{\sum_{i=0}^{k}\binom{k}{i}F_{ni-1}F_{n(k-i)+1} - \sum_{j=1}^{k-1}\binom{k}{j}F_{nj}F_{n(k-j)}}$$

is an integer for all positive integers k and n.

B-818 ***Proposed by L. C. Hsu, Dalian University of Technology, Dalian, China***

Let $H_n = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$. Find a closed form for

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}H_{2k}.$$

B-819 ***Proposed by David Zeitlin, Minneapolis, MN***

Find integers a, b, c, and d (with $1<a<b<c<d$) that make the following an identity:

$$P_n = P_{n-a} + 444P_{n-b} + P_{n-c} + P_{n-d},$$

where P_n is the Pell sequence, defined by $P_{n+2} = 2P_{n+1} + P_n$, for $n \ge 0$, with $P_0 = 0$, $P_1 = 1$.

NOTE: The Elementary Problems Column is in need of more *easy*, yet elegant and nonroutine problems.

SOLUTIONS

Generalizing a Pell Congruence

B-787 ***Proposed by H.-J. Seiffert, Berlin, Germany***
(Vol. 33, no. 2, May 1995)

For $n \ge 0$ and $k > 0$, it is known that F_{kn}/F_k and P_{kn}/P_k are integers. Show that these two integers are congruent modulo $R_k - L_k$.

[**Note:** P_n and $R_n = 2Q_n$ are the Pell and Pell-Lucas numbers, respectively, defined by $P_{n+2} = 2P_{n+1} + P_n$, $P_0 = 0$, $P_1 = 1$ and $Q_{n+2} = 2Q_{n+1} + Q_n$, $Q_0 = 1$, $Q_1 = 1$.]

Solution by Lawrence Somer, Catholic University of America, Washington, DC

We will prove the following more general result. Let $\langle A_n\rangle_{n=0}^{\infty}$ and $\langle C_n\rangle_{n=0}^{\infty}$ denote two second-order linear recurrences satisfying the respective recursion relations

$$A_{n+2} = aA_{n+1} - bA_n, \qquad A_0 = 0, \quad A_1 = 1,$$
$$C_{n+2} = cC_{n+1} - bC_n, \qquad C_0 = 0, \quad C_1 = 1,$$

where a, b, and c are nonzero integers. We assume that $\langle A_n\rangle$ and $\langle C_n\rangle$ are both nondegenerate second-order linear recurrences with $A_n \ne 0$ and $C_n \ne 0$ for any $n \ge 1$.

Let $\langle B_n\rangle$ be a sequence satisfying the same recursion relation as $\langle A_n\rangle$ but having initial terms $B_0 = 2$ and $B_1 = a$. Let $\langle D_n\rangle$ be a sequence satisfying the same recursion relation as $\langle C_n\rangle$ but having initial terms $C_0 = 2$ and $C_1 = c$. Then, for $n \ge 0$ and $k > 0$, we have that A_{kn}/A_k and C_{kn}/C_k are integers and

$$A_{kn}/A_k \equiv C_{kn}/C_k \pmod{B_k - D_k}. \tag{1}$$

Proof: We first note that it is well known that A_{kn}/A_k and C_{kn}/C_k are integers, since $A_kC_k \ne 0$. It was proven ([1], p. 437) that, for a fixed $k \ge 1$, both $\langle A_{kn}/A_k\rangle_{n=0}^{\infty}$ and $\langle C_{kn}/C_k\rangle_{n=0}^{\infty}$ are second-order linear recurrences satisfying the recursion relations

$$r_{n+2} = B_k r_{n+1} - b^k r_n \tag{2}$$

and

$$s_{n+2} = D_k s_{n+1} - b^k s_n, \tag{3}$$

respectively. To establish our result, it suffices to show that (1) holds for k fixed and n varying over the nonnegative integers. We proceed by induction:

$$A_{k\cdot 0} / A_k = 0 \equiv C_{k\cdot 0} / C_k = 0 \pmod{B_k - D_k};$$
$$A_{k\cdot 1} / A_k = 1 \equiv C_{k\cdot 1} / C_k = 1 \pmod{B_k - D_k}.$$

Assume the result holds up to n. By (2) and (3),

$$A_{k(n+1)} / A_k = B_k A_{kn} / A_k - b^k A_{k(n-1)} / A_k \tag{4}$$

and

$$C_{k(n+1)} / C_k = D_k C_{kn} / C_k - b^k C_{k(n-1)} / C_k. \tag{5}$$

Clearly $B_k \equiv D_k \pmod{B_k - D_k}$. Moreover, by our induction hypothesis, $A_{kn} / A_k \equiv C_{kn} / C_k$ and $A_{k(n-1)} / A_k \equiv C_{k(n-1)} / C_k \pmod{B_k - D_k}$. It now follows from (4) and (5) that $A_{k(n+1)} / A_k \equiv C_{k(n+1)} / C_k \pmod{B_k - D_k}$. The result now follows by induction.

If $a = 2$, $b = -1$, and $c = 1$, we get the original problem.

Reference

1. D. H. Lehmer. "An Extended Theory of Lucas' Functions." *Annals of Mathematics* (2) **31** (1930):419-48.

Also solved by Paul S. Bruckman, Andrej Dujella, Pentti Haukkanen, Norbert Jensen, Dorka O. Popova, Tony Shannon, and the proposer.

Asymptotic Analysis

B-788 ***Proposed by Russell Jay Hendel, University of Louisville, Louisville, KY (Vol. 33, no. 2, May 1995)***

(a) Let $G_n = F_{n^2}$. Prove that $G_{n+1} \sim L_{2n+1} G_n$.

(b) Find the error term. More specifically, find a constant C such that $G_{n+1} \sim L_{2n+1} G_n + C G_{n-1}$.

Solution by H.-J. Seiffert, Berlin, Germany

We shall prove that, for all positive integers n, $G_{n+1} = L_{2n+1} G_n + \beta^2 G_{n-1} - \beta^{(n-1)^2}$. Since $\lim_{n\to\infty} \beta^{(n-1)^2} = 0$, this is much more than is asked for in the proposal. In particular, the constant of part (b) is $C = \beta^2 = (3 - \sqrt{5}) / 2$.

Using the Binet formulas, we obtain

$$\begin{aligned}
\sqrt{5}(G_{n+1} - L_{2n+1} G_n - \beta^2 G_{n-1}) &= \sqrt{5}\left(F_{(n+1)^2} - L_{2n+1} F_{n^2} - \beta^2 F_{(n-1)^2}\right) \\
&= \alpha^{(n+1)^2} - \beta^{(n+1)^2} - (\alpha^{2n+1} + \beta^{2n+1})(\alpha^{n^2} - \beta^{n^2}) \\
&\quad - \beta^2(\alpha^{(n-1)^2} - \beta^{(n-1)^2}) \\
&= \alpha^{2n+1}\beta^{n^2} - \beta^{2n+1}\alpha^{n^2} - \beta^2\alpha^{(n-1)^2} + \beta^2\beta^{(n-1)^2} \\
&= -\beta^{n^2-2n-1} + \alpha^{n^2-2n-1} - \alpha^{(n-1)^2-2} + \beta^2\beta^{(n-1)^2} \\
&= \beta^{(n-1)^2}(\beta^2 - 1/\beta^2) = -\sqrt{5}\beta^{(n-1)^2},
\end{aligned}$$

where we have used that $\alpha\beta = -1$. This proves the desired equation.

Also solved by Paul S. Bruckman, Andrej Dujella, Russell Jay Hendel, Norbert Jensen, Can. A. Minh, Tony Shannon, and the proposer.

Differential Equation Involving Lucas Polynomials

B-789 ***Proposed by Richard André-Jeannin, Longwy, France (Vol. 33, no. 2, May 1995)***

The Lucas polynomials, $L_n(x)$, are defined by $L_0 = 2$, $L_1 = x$, and $L_n = xL_{n-1} + L_{n-2}$, for $n \geq 2$.

Find a differential equation satisfied by $L_n^{(k)}$, the k^{th} derivative of $L_n(x)$, where k is a nonnegative integer.

Solution by Andrej Dujella, University of Zagreb, Croatia

The Binet form for $L_n(x)$ is

$$L_n(x) = \left(\frac{x+\sqrt{x^2+4}}{2}\right)^n + \left(\frac{x-\sqrt{x^2+4}}{2}\right)^n.$$

Take the derivative of both sides to get $L_n'(x) = nF_n(x)$.

Combining this with the relation $L_n^2(x) - (x^2+4)F_n^2(x) = 4(-1)^n$, we get the differential equation for $L_n(x)$:

$$(x^2+4)L_n''(x) + xL_n' - n^2 L_n(x) = 0.$$

Taking the derivative k times gives $(x^2+4)L_n^{(k+2)}(x) + (2k+1)xL_n^{(k+1)} - (n^2-k^2)L_n^{(k)} = 0$. Hence, the function $L_n^{(k)}$ satisfies the differential equation

$$(x^2+4)y'' + (2k+1)xy' - (n^2-k^2)y = 0.$$

Also solved by Paul S. Bruckman, Charles K. Cook, Russell Jay Hendel, Can. A. Minh, Igor O. Popov, H.-J. Seiffert, Tony Shannon, M. N. S. Swamy, and the proposer.

Even Inequality

B-790 ***Proposed by H.-J. Seiffert, Berlin, Germany (Vol. 33, no. 4, August 1995)***

Find the largest constant c such that $F_{n+1}^2 > cF_{2n}$ for all even positive integers n.

Solution by L. A. G. Dresel, Reading, England

When n is even, the Binet forms give

$$\frac{F_{n+1}^2}{F_{2n}} = \frac{(\alpha^{n+1}-\beta^{n+1})^2}{\sqrt{5}(\alpha^{2n}-\beta^{2n})} = \frac{\alpha^2}{\sqrt{5}}\frac{(1+\beta^{2n+2})^2}{(1-\beta^{4n})},$$

and since $0 < \beta^2 < 1$, we see that, for all positive even n, $F_{n+1}^2 / F_{2n} > \alpha^2/\sqrt{5}$. As $n \to \infty$, we have $F_{n+1}^2/F_{2n} \to \alpha^2/\sqrt{5}$. It follows that $c = \alpha^2/\sqrt{5} = (5+3\sqrt{5})/10$ is the largest constant such that $F_{n+1}^2 > cF_{2n}$ for all positive even integers n.

Remark: In a similar manner, we can prove the corresponding result for all odd integers n: $F_{n+1}^2 < (\alpha^2/\sqrt{5})F_{2n}$, and that $\alpha^2/\sqrt{5}$ is the smallest constant for which this is true.

Also solved by Charles Ashbacher, Michel A. Ballieu, Paul S. Bruckman, Charles K. Cook, Andrej Dujella, Russell Euler, C. Georghiou, Russell Jay Hendel, Hans Kappus, Carl Libis, Dorka O. Popova, Bob Prielipp, Lawrence Somer, and the proposer.

Divisibility by 18

B-791 ***Proposed by Andrew Cusumano, Great Neck, NY (Vol. 33, no. 4, August 1995)***

Prove that, for all n, $F_{n+11}+F_{n+7}+8F_{n+5}+F_{n+3}+2F_n$ is divisible by 18.

Solution by H.-J. Seiffert, Berlin, Germany

Let $c_0, c_1, \ldots, c_m$ be arbitrary integers. We shall prove that all sums $S_n = \sum_{k=0}^{m} c_k F_{n+k}$, where n is an integer, are divisible by $\gcd(S_0, S_1)$, provided that S_0 and S_1 are not both zero.

It is clear that $S_{n+2} = S_{n+1} + S_n$ for all integers n. It follows that $S_n = S_0 F_{n-1} + S_1 F_n$ for all integers n. The claim follows.

The expression considered in this proposal is such an S_n. Here, $S_0 = 144$ and $S_1 = 234$. The conclusion follows from the fact that $\gcd(144, 234) = 18$.

Haukkanen found the same generalization as Seiffert. Many solvers also came up with the explicit formula $F_{n+11}+F_{n+7}+8F_{n+5}+F_{n+3}+2F_n = 18F_{n+6}$ which makes the result obvious.

Also solved by Charles Ashbacher, Michel A. Ballieu, Glenn Bookhout, Paul S. Bruckman, David M. Burton, Charles K. Cook, Leonard A. G. Dresel, Andrej Dujella, Russell Euler, C. Georghiou, Pentti Haukkanen, Russell Jay Hendel, Joseph J. Koštál, Carl Libis, Can. A. Minh, Bob Prielipp, Don Redmond, R. P. Sealy, Sahib Singh, and the proposer.

Reciprocal Sum

B-792 ***Proposed by Paul S. Bruckman, Edmonds, WA (Vol. 33, no. 4, August 1995)***

Let the sequence $\langle a_n \rangle$ be defined by the recurrence $a_{n+1} = a_n^2 - a_n + 1$, $n > 0$, where the initial term, a_1, is an arbitrary real number larger than 1. Express $\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\cdots$ in terms of a_1.

Solution by Hans Kappus, Rodersdorf, Switzerland

The sum in question is $S = 1/(a_1 - 1)$.

Proof: For $n > 0$, let $S_n = \sum_{k=1}^{n} \frac{1}{a_k}$. The recurrence $a_{k+1} = a_k^2 - a_k + 1$ is equivalent to

$$\frac{1}{a_k} = \frac{1}{a_k - 1} - \frac{1}{a_{k+1} - 1}.$$

Thus, S_n is unmasked as a telescoping sum. Hence,

$$S_n = \frac{1}{a_1 - 1} - \frac{1}{a_{n+1} - 1}.$$

Since it is clear that $\lim_{n\to\infty} a_n = \infty$, the result follows.

Also solved by Andrej Dujella, C. Georghiou, Russell Jay Hendel, Joseph J. Koštál, H.-J. Seiffert, and the proposer.

A Congruence for $2^n L_n$

B-793 *Proposed by Wray Brady, Jalisco, Mexico (Vol. 33, no. 4, August 1995)*

Show that $2^n L_n \equiv 2 \pmod 5$ for all positive integers n.

Solution 1 by Can. A. Minh, University of California at Berkeley, Berkeley, CA

The proof is by complete induction on n. The result is clearly true for $n=1$ and $n=2$. Suppose the assertion holds for $1 \le m < n$; we show it holds for n. We have $2^n L_n = 2^n(L_{n-1}+L_{n-2}) = 2(2^{n-1}L_{n-1})+2^2(2^{n-2}L_{n-2}) \equiv 2\cdot 2 \pmod 5 + 2^2\cdot 2 \pmod 5 \equiv 12 \pmod 5 \equiv 2 \pmod 5$, and we are done.

Solution 2 by H.-J. Seiffert, Berlin, Germany

From the well-known identities [see Problem B-660, this Quarterly **29.1** (1991):86],

$$2^n L_n = 2\sum_{i=0}^{\lfloor n/2 \rfloor}\binom{n}{2i}5^i \quad \text{and} \quad 2^n F_n = 2\sum_{i=1}^{\lfloor (n+1)/2 \rfloor}\binom{n}{2i-1}5^{i-1},$$

it obviously follows that, for all nonnegative integers n,

$$2^n L_n \equiv 2 \pmod{10} \quad \text{and} \quad 2^n F_n \equiv 2n \pmod{10}.$$

Also solved by Charles Ashbacher, Glenn Bookhout, Paul S. Bruckman, David M. Burton, Charles K. Cook, Leonard A. G. Dresel, Andrej Dujella, Russell Euler, C. Georghiou, Russell Jay Hendel, Hans Kappus, Joseph J. Koštál, Carl Libis, Dorka O. Popova, Bob Prielipp, R. P. Sealy, Sahib Singh, Lawrence Somer, and the proposer.

Exponential Inequality

B-794 *Proposed by Zdravko F. Starc, Vršac, Yugoslavia (Vol. 33, no. 4, August 1995)*

For x a real number and n a positive integer, prove that

$$S_n = \left(\frac{F_2}{F_1}\right)^x + \left(\frac{F_3}{F_2}\right)^x + \cdots + \left(\frac{F_{n+1}}{F_n}\right)^x \ge n + x\ln F_{n+1}.$$

Solution by C. Georghiou, University of Patras, Greece

From the Arithmetic-Geometric Mean Inequality, we get $S_n \ge n\left(\frac{F_{n+1}}{F_1}\right)^{x/n} = n\exp\left(\frac{x}{n}\ln F_{n+1}\right) \ge n + x\ln F_{n+1}$, where we have used the inequality $e^y \ge 1+y$, valid for $y \ge 0$.

Also solved by Paul S. Bruckman, Leonard A. G. Dresel, Andrej Dujella, Russell Euler, Hans Kappus, Joseph J. Koštál, Bob Prielipp, H.-J. Seiffert, and the proposer.

Addenda: The following were inadvertantly omitted as solvers of problems presented in previous issues of this Quarterly: Mohammad K. Azarian solved B-780; Andrej Dujella solved B-782–B-783; Russell J. Hendel solved B-785–B-786; Harris Kwong solved B-780–B-781 and B-784–B-785; Igor O. Popov solved B-784 and B-786; R. P. Sealy solved B-778–780 and B-784–B-785.

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