ON THE LIMIT OF GENERALIZED GOLDEN NUMBERS

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1. INTRODUCTION

In this paper we discuss the asymptotic behavior of maximal real roots of generalized Fibonacci polynomials defined recursively by

$$G_{n+2}(x) = xG_{n+1}(x) + G_n(x), \tag{1}$$

for $n \geq 0$, with $G_0(x) = -a$, $G_1(x) = x - a$, where a is a real number.

Very recently, G. A. Moore [2] considered, among other things, the limiting behavior of the maximal real roots of $G_n(x)$ defined by (1), and with $G_0(x) = -1$, $G_1(x) = x - 1$. Let g_n denote the maximal real root of $G_n(x)$ which may be called "the generalized golden numbers" following [1]. G. Moore confirmed an implication of computer analysis that the odd-indexed subsequence of $\{g_n\}$ is monotonically increasing and convergent to 3/2 from below, while the even-indexed subsequence of $\{g_n\}$ is monotonically decreasing and convergent to 3/2 from above. Moreover, it was shown that $\{g_n\}$, $n > 2$, is a sequence of irrational numbers. He also guessed that this result may be generalized in the sense that there exists a real number taking the place of 3/2 for other kinds of Fibonacci polynomial sequences defined by (1) with given $G_0(x)$ and $G_1(x)$.

Here we generalize Moore's result by showing that, for Fibonacci polynomial sequences defined by (1) with $G_0(x) = -a$, $G_1(x) = x - a$, where a is a positive real number, $a(a+2)/(a+1)$ is just the limit of the maximal real roots of $G_n(x)$.

It is noteworthy that the demonstration here is different from Moore's in that it does not rely on the previous knowledge of $\{G_n(x)\}$ on the limit point of g_n. In other words, we shall proceed here in a "deductive" rather than a "confirmative" way.

2. EXISTENCE OF $\{g_n\}$

Let $\{G_n(x)\}$ be defined by (1) with $G_0(x) = -a$, $G_1(x) = x - a$, with $a > 0$. It can be checked easily by induction that each $G_n(x)$ is monic with degree n and constant term $-a$. Therefore, for each $n \geq 1$, $G_n(x)$ will tend to positive infinity for x large enough.

Note that $G_1(a) = 0$, $G_2(a) = -a < 0$, $G_3(a) = -a^2 = aG_2(a) < 0$, $G_4(a) = -a^3 - a \leq aG_3(a) < 0$, by induction; suppose that $G_k(a) \leq aG_{k-1}(a) < 0$ for $k \geq 2$. Then, from (1), $G_{k+1}(a) = aG_k(a) + G_{k-1}(a) < 0$, and the induction is completed. Therefore, for each $n \geq 1$, there exists at least one real root of $G_n(x)$ on $[a, +\infty)$ and, by definition, $g_n > a$.

On the other hand, it can be checked readily using the recursive relation (1) and by an induction argument that we have $G_n(x) > 0$ for $x \in [a+1, +\infty)$.

Therefore, each $G_n(x)$ $(n \geq 2)$ has at least one root on the interval $[a, a+1)$. In particular, $g_n \in [a, a+1)$.

Lemma 2.1:[2] If r is the maximal real root of a function f with positive leading coefficient, then $f(x) > 0$ for all $x > r$. Conversely, if $f(x) > 0$ for all $x \geq t$, then $r < t$. If $f(s) < 0$, then $s < r$.

Remark 2.2: If $a \geq 1$ is an integer, then a standard algebraic argument may be applied to show that the maximal real root of $G_n(x)$ is actually irrational.

3. THE MONOTONICITY OF $\{g_{2n-1}\}$ AND $\{g_{2n}\}$

To illustrate the monotonicity of $\{g_{2n-1}\}$ and $\{g_{2n}\}$, we need a formula of [2] which may be verified by induction.

Formula 3.1: $G_{n+k}(g_n) = (-1)^{k+1}G_{n-k}(g_n)$ for $n \geq k$.

Proposition 3.2: $\{g_{2n-1}\}$ is a monotonically increasing sequence and $\{g_{2n}\}$ is a monotonically decreasing sequence. Moreover, $g_{2n-1} < g_{2n}$.

Proof:

Odd-Indexed Sequence. It can be checked readily that $G_3(a) = -a^2 < 0$; thus, $g_3 > a = g_1$. Assume that, by induction, $g_1 < g_3 < \cdots < g_{2k-3} < g_{2k-1}$. Then, by Lemma 2.1, $G_{2k-3}(g_{2k-1}) > G_{2k-3}(g_{2k-3}) = 0$. Using Formula 3.1, we get

$$G_{2k+1}(g_{2k-1}) = G_{(2k-1)+2}(g_{2k-1}) = (-1)^3 G_{(2k-1)-2}(g_{2k-1}) < 0.$$

It follows from Lemma 2.1 that G_{2k+1} has a real root greater than g_{2k-1}, and thus $g_{2k-1} < g_{2k+1}$.

Even-Indexed Sequence. Using recursive formula (1), one obtains

$$g_{2k+1}(g_{2k-1}) = g_{2k-1}G_{2k}(g_{2k-1}) + G_{2k-1}(g_{2k-1}) = g_{2k-1}G_{2k}(g_{2k-1}).$$

Since $g_{2k-1} < g_{2k}$, it follows from Lemma 2.1 that $G_{2k-1}(g_{2k}) > 0$, thus $G_{2k-2}(g_{2k}) < 0$, and it follows from Lemma 2.1 again that $g_{2k} < g_{2k-2}$. □

4. THE CONVERGENCE OF g_{2n-1} AND g_{2n}

It is known now that $\{g_{2n-1}\}$ is monotonically increasing, bounded above by $a+1$, and $\{g_{2n}\}$ is monotonically decreasing, bounded below by a. Thus, limits exist for both of the sequences. Denote by $g_{\text{odd}} =: \lim_{n\to\infty} g_{2n-1}$, $g_{\text{even}} =: \lim_{n\to\infty} g_{2n}$. Then Proposition 3.2 implies $g_{\text{odd}} \leq g_{\text{even}}$. Our aim here is to show that $g_{\text{odd}} = g_{\text{even}}$, which is included in the following theorem.

Theorem 4.1: Both $\{g_{2n-1}\}$ and $\{g_{2n}\}$ converge to $\xi = a(a+2)/(a+1)$ when n tends to infinity.

Remark 4.2: If a is an integer, then, from Proposition 3.2, g_n is a sequence of irrationals that converges to a rational number $a(a+2)/(a+1)$. This reduces to Moore's result in [2] when $a = 1$.

Proof: Since $G_n(x)$ may be expressed in terms of roots of its characteristic equation as

$$G_n(x) = C_1(x)\lambda_1(x)^n + C_2(x)\lambda_2(x)^n, \tag{2}$$

where

$$\lambda_1(x) = \frac{x+\sqrt{x^2+4}}{2} > \lambda_2(x) = \frac{x-\sqrt{x^2+4}}{2}, \tag{3}$$

and

$$
\begin{aligned}
C_1(x) &= \left[2(x-a)+ax-a\sqrt{x^2+4}\right]/2\sqrt{x^2+4}, \\
C_2(x) &= \left[-2(x-a)-ax-a\sqrt{x^2+4}\right]/2\sqrt{x^2+4}.
\end{aligned} \tag{4}
$$

It is seen readily that $\lambda_1(x) \geq \lambda_1(a) > 1$, $|\lambda_2(x)| = 1/\lambda_1(x) \leq 1/\lambda_1(a)$ for $x \in [a, a+1]$. Therefore, $\lim_{n\to\infty} \lambda_1(x)^n = +\infty$, $\lim_{n\to\infty} \lambda_2(x)^n = 0$ uniformly for $x \in [a, a+1]$.

Now, setting $n = 2k-1$ and $x = g_{2k-1}$ in (2), we obtain

$$C_1(g_{2k-1})\lambda_1(g_{2k-1})^{2k-1} + C_2(g_{2k-1})\lambda_2(g_{2k-1})^{2k-1} = 0.$$

Since $C_1(x)$ and $C_2(x)$ are continuous on the interval $[a, a+1]$, this implies that $|C_1(x)|$ and $|C_2(x)|$ are bounded below and above on $[a, a+1]$. Therefore, we have

$$\lim_{k\to\infty} C_1(g_{2k-1}) = C_1(g_{\text{odd}}) = 0,$$

and it follows that $g_{\text{odd}} = \lim_{k\to\infty} g_{2k-1} = a(a+2)/(a+1)$ by the continuity of $C_1(x)$.

On the other hand, by taking $n = 2k$ in (2), a similar argument can be applied to show that $g_{\text{even}} = a(a+2)/(a+1)$. □

Note that $\lim_{n\to\infty} g_n = 1$ if and only if $a = (\sqrt{5}-1)/2$, the original golden number.

In conclusion, we remark that it may be shown easily that the maximal real root of $G_n'(x)$, denoted by g_n', also exists on the interval $(a, a+1)$ for $n \geq 4$. It seems, from numerical analysis, that the sequence $\{g_n'\}$ is monotonically increasing and converges to $\xi = a(a+2)/(a+1)$. This implication deserves further exposition.

REFERENCES

1. G. Moore. "A Fibonacci Polynomial Sequence Defined by Multidimensional Continued Fractions; and Higher-Order Golden Ratios." *The Fibonacci Quarterly* **31.4** (1993):354-64.
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