MULTIVARIATE SYMMETRIC IDENTITIES

Abdelhamid Abderrezzak
L.I.T.P., Universite Paris, 2 Place Jussieu, 75251 Paris cedex 05 France
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1. INTRODUCTION

As an application of Lagrange inversion, Riordan [9] gave the following expansions:

$$\exp(bz)=\sum_{k=0}^{\infty}\frac{b(ak+b)^{k-1}}{k!}(z\exp(-az))^k; \tag{1.1}$$

$$\frac{\exp(bz)}{1-az}=\sum_{k=0}^{\infty}\frac{(ak+b)^k}{k!}(z\exp(-az))^k; \tag{1.2}$$

$$(1-z)^b=\sum_{k=0}^{\infty}\frac{b}{ak+b}\binom{ak+b}{k}\left(\frac{z}{(1+z)^a}\right)^k; \tag{1.3}$$

$$\frac{(1+z)^b}{1-\frac{az}{1+z}}=\sum_{k=0}^{\infty}\binom{ak+b}{k}\left(\frac{z}{(1+z)^a}\right)^k. \tag{1.4}$$

Starting from these identities, Gould ([3], [4]) obtained various convolution identities. The multivariate case of (1.2) and (1.4) was obtained by Carlitz [1] using MacMahon's "master theorem."

Using other methods, Cohen and Hudson [2] gave bivariate generalizations of (1.1) and (1.2) that are different from those of Carlitz.

Krattenthaler [5] showed that the preceding formulas are a consequence of his bivariate version of Lagrange inversion; furthermore, he has generalized (1.3) and (1.4).

One must note that we do not need to use Lagrange inversion in two variables to prove these types of identities, as Krattenthaler did, but need only use Lagrange interpolation, which is a much simpler tool.

Lagrange interpolation must be considered as describing the properties of a linear operator sending a function of one variable to a symmetric function. It can be written as a summation on a set or as a product of divided differences; it is this latter version that we shall use here. In fact, in Section 2 we give the four Lagrange interpolation formulas, (2.1)-(2-4), that contain many of Krattenthaler's identities as special cases. In Section 3 we show how our Lagrange interpolation formulas can even be used to derive q-analogs of these identities.

2. MULTIPLE INTERPOLATION

Let $A=\{a_1,a_2,a_3,\dots\}$ and $B=\{b_1,b_2,b_3,\dots\}$ be two alphabets and let κ, ϕ, and ζ be three bivariate functions of x and y. For all positive integers m and n, put

$$K_1(m,n,x,y)=\kappa(x,y)(\phi(x,y))^m(\zeta(x,y))^n,$$
$$K_2(m,n,x,y)=\kappa(x,y)(\phi(x,y))_m(\zeta(x,y))_n,$$
$$K_3(m,n,x,y)=\kappa(x,y)(\phi(x,y))^m(\zeta(x,y))_n,$$
$$K_4(m,n,x,y)=\kappa(x,y)(\phi(x,y))_m(\zeta(x,y))^n,$$

where $(a)\ln = a(a-1)(a-2)\cdots(a-n+1)$.

Lagrange interpolation generally stated that, for any function of one variable x, and "interpolation points" b, c, d, ...

$$\frac{f(x)}{(x-b)(x-c)(x-d)\cdots}=\frac{f(b)}{(x-b)(b-c)(b-d)\cdots}+\frac{f(c)}{(x-c)(c-b)(c-d)\cdots}$$
$$+\frac{f(d)}{(x-d)(d-b)(d-c)\cdots}+\textit{remainder}.$$

We shall only need the Lagrange interpolation formula, but written in a symmetrical manner. It is more satisfactory to consider the set $A=\{x,b,c,\dots,d\}$ and write

$$\sum_{a\in A}\frac{f(a)}{R(a,\,A\setminus a)}=\textit{remainder},$$

where $R(a,\,A\setminus a)$ is the product $\prod_{a'\neq a}(a-a')$.

In other words, Lagrange interpolation amounts to considering properties of the linear operator $f\to\sum_{a\in A}f(a)/R(a,\,A\setminus a)$. This operator sends a polynomial of degree k to a symmetric polynomial in A of degree $k-n$, with $\mathrm{card}(A)=n+1$. In particular, it annihilates polynomials of degree $<n$, and maps $f(x)=x^n$ to the constant 1.

These properties suffice to characterize the Lagrange operator.

If $\phi(x,y,z,\dots)$ is a polynomial, the difference $\phi(x,y,z,\dots)-\phi(y,x,z,\dots)$ is divisible by $x-y$. Following Newton, for any pair of variables (x,y), one defines a divided difference operator, ∂_{xy}, acting on the ring of polynomials as

$$\phi(x,y,\dots)\to\partial_{xy}\phi(x,y,\dots)=\frac{\phi(x,y,z,\dots)-\phi(y,x,z,\dots)}{x-y}.$$

It is clear that the product (now we need to order $A_{n+1}:=\{a_1,a_2,\dots,a_{n+1}\}$)

$$\Lambda(A_{m+1}):=\partial_{a_n,a_{n+1}}\cdot\partial_{a_{n-1},a_n}\cdot\cdots\cdot\partial_{a_1,a_2}$$

also satisfies the same properties and, therefore, coincides with the Lagrange operator (see [7]). Thus, we have

$$\Lambda(A_{m+1})\phi(a_1)=\sum_{k=1}^{m}\frac{\phi(a_{k+1})}{R(a_{k+1},\,A_{m+1}\setminus a_{k+1})}.$$

One can note that divided differences are also the main ingredient in the Newton interpolation formula, and by relating their properties to the symmetric group one can extend Newton interpolation to multivariable functions (see [6] and [8]).

For our purpose, we shall use Lagrange interpolation for two independent alphabets and functions of two variables:

$$\Lambda(A_{m+1})\Lambda(B_{n+1})\phi(a_1,b_1)=\sum_{k=0}^{m}\sum_{p=0}^{n}\frac{\phi(a_{k+1},b_{p+1})}{R(a_{k+1},A_{m+1}\setminus a_{k+1})R(b_{p+1},B_{n+1}\setminus b_{p+1})}.$$

We deduce, without difficulties, the following theorem.

Theorem: Let $A=\{a_1,a_2,a_3,\dots\}$ and $B=\{b_1,b_2,b_3,\dots\}$ be two infinite alphabets. Then we have:

$$\begin{aligned}&\sum_{m\geq 0,n\geq 0}\Lambda(A_{m+1})\Lambda(B_{n+1})K_1(m,n,a_1,b_1)t^mz^n\\ &=\sum_{k\geq 0,\,p\geq 0}\kappa(a_{k+1},b_{p+1})\frac{(\phi(a_{k+1},b_{p+1}))^k(\zeta(a_{k+1},b_{p+1}))^p}{R(a_{k+1},A_k)R(b_{p+1},B_p)}t^kz^p\\ &\times\sum_{m\geq 0,n\geq 0}\frac{(\phi(a_{k+1},b_{p+1}))^m(\zeta(a_{k+1},b_{p+1}))^n}{R(a_{k+1},A_{m+k+1}\setminus A_{k+1})R(b_{p+1},B_{n+p+1}\setminus B_{p+1})}t^mz^n;\end{aligned}\tag{2.1}$$

$$\begin{aligned}&\sum_{m\geq 0,n\geq 0}\Lambda(A_{m+1})\Lambda(B_{n+1})K_2(m,n,a_1,b_1)t^mz^n\\ &=\sum_{k\geq 0,\,p\geq 0}\kappa(a_{k+1},b_{p+1})\frac{(\phi(a_{k+1},b_{p+1}))_k(\zeta(a_{k+1},b_{p+1}))_p}{R(a_{k+1},A_k)R(b_{p+1},B_p)}t^kz^p\\ &\times\sum_{m\geq 0,n\geq 0}\frac{(\phi(a_{k+1},b_{p+1}))_m(\zeta(a_{k+1},b_{p+1}))_n}{R(a_{k+1},A_{m+k+1}\setminus A_{k+1})R(b_{p+1},B_{n+p+1}\setminus B_{p+1})}t^mz^n;\end{aligned}\tag{2.2}$$

$$\begin{aligned}&\sum_{m\geq 0,n\geq 0}\Lambda(A_{m+1})\Lambda(B_{n+1})K_3(m,n,a_1,b_1)t^mz^n\\ &=\sum_{k\geq 0,\,p\geq 0}\kappa(a_{k+1},b_{p+1})\frac{(\phi(a_{k+1},b_{p+1}))^k(\zeta(a_{k+1},b_{p+1}))_p}{R(a_{k+1},A_k)R(b_{p+1},B_p)}t^kz^p\\ &\times\sum_{m\geq 0,n\geq 0}\frac{(\phi(a_{k+1},b_{p+1}))^m(\zeta(a_{k+1},b_{p+1}))_n}{R(a_{k+1},A_{m+k+1}\setminus A_{k+1})R(b_{p+1},B_{n+p+1}\setminus B_{p+1})}t^mz^n;\end{aligned}\tag{2.3}$$

$$\begin{aligned}&\sum_{m\geq 0,n\geq 0}\Lambda(A_{m+1})\Lambda(B_{n+1})K_4(m,n,a_1,b_1)t^mz^n\\ &=\sum_{k\geq 0,\,p\geq 0}\kappa(a_{k+1},b_{p+1})\frac{(\phi(a_{k+1},b_{p+1}))_k(\zeta(a_{k+1},b_{p+1}))^p}{R(a_{k+1},A_k)R(b_{p+1},B_p)}t^kz^p\\ &\times\sum_{m\geq 0,n\geq 0}\frac{(\phi(a_{k+1},b_{p+1}))_m(\zeta(a_{k+1},b_{p+1}))^n}{R(a_{k+1},A_{m+k+1}\setminus A_{k+1})R(b_{p+1},B_{n+p+1}\setminus B_{p+1})}t^mz^n.\end{aligned}\tag{2.4}$$

We shall use the above theorem in the case of different specializations κ, ϕ, and ζ for which the divided difference is easily calculable. The simple fact that the operator $\Lambda(A_{m+1})$ decreases the total degree in A by m implies the following identities.

Lemma (2.5): If we specialize $\kappa(x,y)\to 1$, $\phi\to\phi_1(x,y)=\frac{\lambda_1+\mu_1 x}{\lambda_2+\mu_2 y}$ and $\zeta\to\zeta_1(x,y)=\frac{\lambda_2+\mu_2 y}{\lambda_1+\mu_1 x}$, we obtain

$$\begin{aligned}\Lambda(A_{m+1})\Lambda(B_{n+1})K_1(m,n,a_1,b_1)&=\Lambda(A_{m+1})\Lambda(B_{n+1})K_2(m,n,a_1,b_1)\\&=\Lambda(A_{m+1})\Lambda(B_{n+1})K_3(m,n,a_1,b_1)\\&=\begin{cases}\left(\frac{\mu_1}{\lambda_2+\mu_2 b_1}\right)^m & \text{if } n=0,\\ \left(\frac{\mu_2}{\lambda_1+\mu_1 a_1}\right)^n & \text{if } m=0,\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

Lemma (2.6): If we put $\kappa(x,y)\to\kappa_1(x,y)=\frac{1}{\lambda_1+\mu_1 x}$, $\phi\to\phi_1(x,y)=\frac{\lambda_1+\mu_1 x}{\lambda_2+\mu_2 y}$, and $\zeta\to\zeta_2(x,y)=\frac{\lambda_2+\mu_2 y}{\lambda_1+\mu_1 x}(\alpha_1+\beta_1 x)$, we obtain

$$\begin{aligned}\Lambda(A_{m+1})\Lambda(B_{n+1})K_1(m,n,a_1,b_1)&=\Lambda(A_{m+1})\Lambda(B_{n+1})K_2(m,n,a_1,b_1)\\&=\Lambda(A_{m+1})\Lambda(B_{n+1})K_3(m,n,a_1,b_1)\\&=\Lambda(A_{m+1})\Lambda(B_{n+1})K_4(m,n,a_1,b_1)\\&=\begin{cases}\frac{1}{\lambda_1+\mu_1 a_1}\left(\frac{\beta_1+\alpha_1 a_1}{\lambda_1+\mu_1 a_1}\right)^m & \text{if } m=0,\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

Lemma (2.7): If we put $\kappa(x,y)\to\kappa_2(x,y)=\frac{\lambda_1\alpha_2}{(\lambda_1+\mu_1 x)(\alpha_2+\beta_{2y})}+\frac{\lambda_2\alpha_1}{(\lambda_2+\mu_2 y)(\alpha_1+\beta_1 x)}-\frac{\alpha_1\alpha_2}{(\alpha_1+\beta_1 x)(\alpha_2+\beta_2 y)}$, $\phi\to\phi_2(x,y)=\frac{\lambda_1+\mu_1 x}{\lambda_2+\mu_2 y}(\alpha_2+\beta_2 y)$, and $\zeta\to\zeta_2(x,y)=\frac{\lambda_2+\mu_2 y}{\lambda_1+\mu_1 x}(\alpha_1+\beta_1 x)$, we obtain

$$\begin{aligned}\Lambda(A_{m+1})\Lambda(B_{n+1})K_1(m,n,a_1,b_1)&=\Lambda(A_{m+1})\Lambda(B_{n+1})K_2(m,n,a_1,b_1)\\&=\Lambda(A_{m+1})\Lambda(B_{n+1})K_3(m,n,a_1,b_1)\\&=\begin{cases}\kappa_2(a_1,b_1) & \text{if } m=n=0,\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

Lemma (2.8): If we put $\kappa(x,y)\to\kappa_3(x,y)=\frac{1}{(\lambda_1+\mu_1 x)(\lambda_2+\mu_2 y)}$, $\phi\to\phi_2(x,y)=\frac{\lambda_1+\mu_1 x}{\lambda_2+\mu_2 y}(\alpha_2+\beta_2 y)$, and $\zeta\to\zeta_2(x,y)=\frac{\lambda_2+\mu_2 y}{\lambda_1+\mu_1 x}(\alpha_1+\beta_1 x)$, we obtain

$$\begin{aligned}&\Lambda(A_{m+1})\Lambda(B_{n+1})K_1(m,n,a_1,b_1)\\&=\begin{cases}\frac{1}{\mu_1\mu_2}\left(\alpha_1-\frac{\beta_1}{\mu_1}\lambda_1\right)^m\left(\alpha_2-\frac{\beta_2}{\mu_2}\lambda_2\right)^n\left(\prod_{j=0}^{m}\left(\frac{\lambda_1}{\mu_1}+a_{j+1}\right)\right)^{-1}\left(\prod_{j=0}^{n}\left(\frac{\lambda_2}{\mu_2}+b_{j+1}\right)\right)^{-1} & \text{if } m=n,\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

The identities (2.1)-(2.10) and (3.16) of Krattenthaler [5] arise as different specializations of the functions κ,ϕ, and ζ considered in Lemmas (2.5)-(2.8) above, and to the case in which $A=B=\{0,1,2,\dots\}$. For each of the four cases given in our Theorem, we give the formulas when A and B are general. We then specialize to the case where A and B are sequences of "q-integers."

3. APPLICATION OF IDENTITY (2.1)

Formula (2.1) and the previous lemmas provide the following identities:

I) In the case where A and B are general alphabets, we have

$$\begin{aligned}&\left(1-\frac{\mu_1}{\lambda_1+\mu_1 a_1\lambda_2+\mu_2 b_1}tz\right)\left(1-\frac{\mu_1}{\lambda_2+\mu_2 b_1}t\right)^{-1}\left(1-\frac{\mu_2}{\lambda_1+\mu_1 a_1}z\right)^{-1}\\ &=\sum_{k\geq 0,\ p\geq 0}\frac{(\phi_1(a_{k+1},b_{p+1}))^k(\zeta_1(a_{k+1},b_{p+1}))^p}{R(a_{k+1},A_k)R(b_{p+1},B_p)}t^k z^p\\ &\times\sum_{m\geq 0,\ n\geq 0}\frac{(\phi_1(a_{k+1},b_{p+1}))^m(\zeta_1(a_{k+1},b_{p+1}))^n}{R(a_{k+1},A_{m+k+1}\setminus A_{k+1})R(b_{p+1},B_{n+p+1}\setminus B_{p+1})}t^m z^n,\end{aligned} \tag{3.1}$$

$$\begin{aligned}\frac{1}{(\lambda_1+\mu_1 a_1)-(\alpha_1+\beta_1 a_1)\mu_2 z}&=\sum_{k\geq 0,\ p\geq 0}\frac{1}{\lambda_1+\mu_1 a_{k+1}}\frac{(\phi_1(a_{k+1},b_{p+1}))^k(\zeta_2(a_{k+1},b_{p+1}))^p}{R(a_{k+1},A_k)R(b_{p+1},B_p)}t^k z^p\\ &\times\sum_{m\geq 0,\ n\geq 0}\frac{(\phi_1(a_{k+1},b_{p+1}))^m(\zeta_2(a_{k+1},b_{p+1}))^n}{R(a_{k+1},A_{m+k+1}\setminus A_{k+1})R(b_{p+1},B_{n+p+1}\setminus B_{p+1})}t^m z^n,\end{aligned} \tag{3.2}$$

$$\begin{aligned}\kappa_2(a_1,b_1)&=\sum_{k\geq 0,\ p\geq 0}\kappa_2(a_{k+1},b_{p+1})\frac{(\phi_2(a_{k+1},b_{p+1}))^k(\zeta_2(a_{k+1},b_{p+1}))^p}{R(a_{k+1},A_k)R(b_{p+1},B_p)}t^k z^p\\ &\times\sum_{m\geq 0,\ n\geq 0}\frac{(\phi_2(a_{k+1},b_{p+1}))^m(\zeta_2(a_{k+1},b_{p+1}))^n}{R(a_{k+1},A_{m+k+1}\setminus A_{k+1})R(b_{p+1},B_{n+p+1}\setminus B_{p+1})}t^m z^n,\end{aligned} \tag{3.3}$$

$$\begin{aligned}&\frac{1}{\mu_1\mu_2}\sum_{j=0}\left(\left(\alpha_1-\frac{\beta_1}{\mu_1}\lambda_1\right)\left(\alpha_2-\frac{\beta_2}{\mu_2}\lambda_2\right)tz\right)^j\left(\prod_{s=0}^{j}\left(\frac{\lambda_1}{\mu_1}+a_{s+1}\right)\right)^{-1}\left(\prod_{s=0}^{j}\left(\frac{\lambda_2}{\mu_2}+b_{s+1}\right)\right)^{-1}\\ &=\sum_{k\geq 0,\ p\geq 0}\frac{1}{(\lambda_1+\mu_1 a_{k+1})(\lambda_2+\mu_2 b_{p+1})}\frac{(\phi_2(a_{k+1},b_{p+1}))^k(\zeta_2(a_{k+1},b_{p+1}))^p}{R(a_{k+1},A_k)R(b_{p+1},B_p)}(-t)^k(-z)^p\\ &\times\sum_{m\geq 0,\ n\geq 0}\frac{(\phi_2(a_{k+1},b_{p+1}))^m(\zeta_2(a_{k+1},b_{p+1}))^n}{R(a_{k+1},A_{m+k+1}\setminus A_{k+1})R(b_{p+1},B_{n+p+1}\setminus B_{p+1})}(-t)^m(-z)^n.\end{aligned} \tag{3.4}$$

II) Let $[n]=\frac{1-q^n}{1-q}$, $[n]!=[n][n-1]\cdots[1]$, $\exp_q(x)=\sum_{n=0}^{\infty}\frac{x^n}{[n]!}$. Then in the case $A=\{0,[1],[2],\ldots\}$, $B=\{0,[1],[2],\ldots\}$, we obtain q-analogs for Krattenthaler's identities (2.1), (2.4), (2.8), and (3.6) in [5]. For example, from (3.1), we obtain the following q-analog of (2.1) in [5].

$$\begin{aligned}\left(1-\frac{\mu_1}{\lambda_1}\frac{\mu_2}{\lambda_2}tz\right)\left(1-\frac{\mu_1}{\lambda_2}t\right)^{-1}\left(1-\frac{\mu_2}{\lambda_1}z\right)^{-1}&=\sum_{k\geq 0,\ p\geq 0}\frac{(\phi_1([k],[p]))^k}{q^{k(k-1)/2}[k]!}\frac{((\zeta_1([k],[p]))^p}{q^{p(p-1)/2}[p]!}t^k z^p\\ &\times\exp_q\left(-\frac{\phi_1([k],[p])}{q^k}t\right)\exp_q\left(-\frac{\zeta_1([k],[p])}{q^p}z\right).\end{aligned} \tag{3.5}$$

4. APPLICATION OF IDENTITY (2.2)

I) In the case where A and B are general alphabets, we have

$$\left(1-\frac{\mu_1}{\lambda_1+\mu_1 a_1}\frac{\mu_2}{\lambda_2+\mu_2 b_1}tz\right)\left(1-\frac{\mu_1}{\lambda_2+\mu_2 b_1}t\right)^{-1}\left(1-\frac{\mu_2}{\lambda_1+\mu_1 a_1}z\right)^{-1}$$
$$=\sum_{k\geq 0,\, p\geq 0}\frac{(\phi_1(a_{k+1},b_{p+1}))_k(\zeta_1(a_{k+1},b_{p+1}))_p}{R(a_{k+1},A_k)R(b_{p+1},B_p)}t^k z^p \tag{4.1}$$
$$\times\sum_{m\geq 0,\, n\geq 0}\frac{(\phi_1(a_{k+1},b_{p+1}))_m(\zeta_1(a_{k+1},b_{p+1}))_n}{R(a_{k+1},A_{m+k+1}\setminus A_{k+1})R(b_{p+1},B_{n+p+1}\setminus B_{p+1})}t^m z^n,$$

$$\frac{1}{(\lambda_1+\mu_1 a_1)-(\alpha_1+\beta_1 a_1)\mu_2 z}=\sum_{k\geq 0,\, p\geq 0}\frac{1}{\lambda_1+\mu_1 a_{k+1}}\frac{(\phi_1(a_{k+1},b_{p+1}))_k(\zeta_2(a_{k+1},b_{p+1}))_p}{R(a_{k+1},A_k)R(b_{p+1},B_p)}t^k z^p$$
$$\times\sum_{m\geq 0,\, n\geq 0}\frac{(\phi_1(a_{k+1},b_{p+1}))_m(\zeta_2(a_{k+1},b_{p+1}))_n}{R(a_{k+1},A_{m+k+1}\setminus A_{k+1})R(b_{p+1},B_{n+p+1}\setminus B_{p+1})}t^m z^n, \tag{4.2}$$

$$\kappa_2(a_1,b_1)=\sum_{k\geq 0,\, p\geq 0}\kappa_2(a_{k+1},b_{p+1})\frac{(\phi_2(a_{k+1},b_{p+1}))_k(\zeta_2(a_{k+1},b_{p+1}))_p}{R(a_{k+1},A_k)R(b_{p+1},B_p)}t^k z^p$$
$$\times\sum_{m\geq 0,\, n\geq 0}\frac{(\phi_2(a_{k+1},b_{p+1}))_m(\zeta_2(a_{k+1},b_{p+1}))_n}{R(a_{k+1},A_{m+k+1}\setminus A_{k+1})R(b_{p+1},B_{n+p+1}\setminus B_{p+1})}t^m z^n, \tag{4.3}$$

II) For example, in the case $A=\{0,[1],[2],\dots\}$, $B=\{0,[1],[2],\dots\}$ we obtain from (4.2) the following q-analog of (2.5) in [5].

$$\frac{1}{\lambda_1-\alpha_1\mu_2 z}=\sum_{k\geq 0,\, p\geq 0}\frac{1}{\lambda_1+\mu_1[k]}\frac{(\phi_1([k],[p]))_k}{q^{k(k-1)/2}[k]!}\frac{((\zeta_2([k],[p]))_p}{q^{p(p-1)/2}[p]!}t^k z^p$$
$$\times\sum_{m\geq 0,\, n\geq 0}(-1)^{m+n}\frac{(\phi_1([k],[p])-k)_m}{q^{mk}[m]!}\frac{(\zeta_2([k],[p])-p)_n}{q^{np}[n]!}t^m z^n. \tag{4.4}$$

5. APPLICATION OF IDENTITY (2.3)

I) In the case where A and B are general alphabets, we have

$$\left(1-\frac{\mu_1}{\lambda_1+\mu_1 a_1}\frac{\mu_2}{\lambda_2+\mu_2 b_1}tz\right)\left(1-\frac{\mu_1}{\lambda_2+\mu_2 b_1}t\right)^{-1}\left(1-\frac{\mu_2}{\lambda_1+\mu_1 a_1}z\right)^{-1}$$
$$=\sum_{k\geq 0,\, p\geq 0}\frac{(\phi_1(a_{k+1},b_{p+1}))^k(\zeta_1(a_{k+1},b_{p+1}))_p}{R(a_{k+1},A_k)R(b_{p+1},B_p)}t^k z^p \tag{5.1}$$
$$\times\sum_{m\geq 0,\, n\geq 0}\frac{(\phi_1(a_{k+1},b_{p+1}))^m(\zeta_1(a_{k+1},b_{p+1}))_n}{R(a_{k+1},A_{m+k+1}\setminus A_{k+1})R(b_{p+1},B_{n+p+1}\setminus B_{p+1})}t^m z^n,$$

$$\frac{1}{(\lambda_1+\mu_1 a_1)-(\alpha_1+\beta_1 a_1)\mu_2 z}=\sum_{k\geq 0,\, p\geq 0}\frac{1}{\lambda_1+\mu_1 a_{k+1}}\frac{(\phi_1(a_{k+1},b_{p+1}))^k(\zeta_2(a_{k+1},b_{p+1}))_p}{R(a_{k+1},A_k)R(b_{p+1},B_p)}t^k z^p$$
$$\times\sum_{m\geq 0,\, n\geq 0}\frac{(\phi_1(a_{k+1},b_{p+1}))^m(\zeta_2(a_{k+1},b_{p+1}))_n}{R(a_{k+1},A_{m+k+1}\setminus A_{k+1})R(b_{p+1},B_{n+p+1}\setminus B_{p+1})}t^m z^n, \tag{5.2}$$

$$\kappa_2(a_1,b_1)=\sum_{k\geq 0,\, p\geq 0}\kappa_2(a_{k+1},b_{p+1})\frac{(\phi_2(a_{k+1},b_{p+1}))^k(\zeta_2(a_{k+1},b_{p+1}))_p}{R(a_{k+1},A_k)R(b_{p+1},B_p)}t^k z^p$$
$$\times\sum_{m\geq 0,\, n\geq 0}\frac{(\phi_2(a_{k+1},b_{p+1}))^m(\zeta_2(a_{k+1},b_{p+1}))_n}{R(a_{k+1},A_{m+k+1}\setminus A_{k+1})R(b_{p+1},B_{n+p+1}\setminus B_{p+1})}t^m z^n, \tag{5.3}$$

II) For example, in the case $A=\{0,[1],[2],\dots\}$, $B=\{0,[1],[2],\dots\}$ we obtain from (5.3) the following q-analog of (2.10) in [5].

$$1=\sum_{k\geq 0,\, p\geq 0}\kappa_2([k],[p])\frac{(\phi_2([k],[p]))^k}{q^{k(k-1)/2}[k]!}\frac{((\zeta_2([k],[p]))_p}{q^{p(p-1)/2}[p]!}t^k z^p$$
$$\times\exp_q\left(-\frac{\phi_2([k],[p]))}{q^k}t\right)\sum_{n\geq 0}(-1)^n\frac{(\zeta_2([k],[p])-p)_n}{q^{np}[n]!}z^n. \tag{5.4}$$

6. APPLICATION OF IDENTITY (2.4)

I) In the case where A and B are general alphabets, we have

$$\frac{1}{(\lambda_1+\mu_1 a_1)-(\alpha_1+\beta_1 a_1)\mu_2 z}=\sum_{k\geq 0,\, p\geq 0}\frac{1}{\lambda_1+\mu_1 a_{k+1}}\frac{(\phi_1(a_{k+1},b_{p+1}))_k(\zeta_2(a_{k+1},b_{p+1}))^p}{R(a_{k+1},A_k)R(b_{p+1},B_p)}t^k z^p$$
$$\times\sum_{m\geq 0,\, n\geq 0}\frac{(\phi_1(a_{k+1},b_{p+1}))_m(\zeta_2(a_{k+1},b_{p+1}))^n}{R(a_{k+1},A_{m+k+1}\setminus A_{k+1})R(b_{p+1},B_{n+p+1}\setminus B_{p+1})}t^m z^n. \tag{6.1}$$

II) For example, in the case $A=\{0,[1],[2],\dots\}$, $B=\{0,[1],[2],\dots\}$ we obtain from (6.1) the following q-analog of (2.7) in [5].

$$\frac{1}{\lambda_1-\alpha_1\mu_2 z}=\sum_{k\geq 0,\, p\geq 0}\frac{1}{\lambda_1+\mu_1[k]}\frac{(\phi_1([k],[p]))_k}{q^{k(k-1)/2}[k]!}\frac{((\zeta_2([k],[p]))^p}{q^{p(p-1)/2}[p]!}t^k z^p$$
$$\times\exp_q\left(-\frac{\zeta_2([k],[p]))}{q^k}z\right)\sum_{m\geq 0}(-1)^m\frac{(\phi_1([k],[p])-p)_m}{q^{mp}[n]!}t^m. \tag{6.2}$$

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