ADVANCED PROBLEMS AND SOLUTIONS

Edited by **Raymond E. Whitney**

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-517 ***Proposed by Paul S. Bruckman, Seattle, WA***

Given a positive integer n, define the sums $P(n)$ and $Q(n)$ as follows:

$$P(n)=\sum_{d\mid n}\mu\left(\frac{n}{d}\right)L_d,\qquad Q(n)=\sum_{d\mid n}\Phi\left(\frac{n}{d}\right)L_d,$$

where μ and Φ and the Möbius and Euler functions, respectively. Show that $n\mid P(n)$ and $n\mid Q(n)$.

H-518 ***Proposed by H.-J. Seiffert, Berlin, Germany***

Define the Fibonacci polynomials by $F_0(x)=0$, $F_1(x)=1$, $F_n(x)=xF_{n-1}(x)+F_{n-2}(x)$, for $n\geq 2$. Show that, for all complex numbers x and y and all positive integers n,

$$\sum_{k=0}^{n}\binom{2n}{n-k}F_k(x)F_k(y)=(x-y)^{n-1}F_n\left(\frac{xy+4}{x-y}\right).\tag{1}$$

As special cases of (1), obtain the following identities:

$$\sum_{\substack{k=0\\ 5\nmid 2n-k-1}}^{2n-1}(-1)^{[(2n-k+1)/5]}\binom{4n-2}{k}=5^{n-1}L_{2n-1};\tag{2}$$

$$\sum_{\substack{k=0\\ 5\nmid 2n-k}}^{2n}(-1)^{[(2n-k+2)/5]}\binom{4n}{k}=5^{n}F_{2n};\tag{3}$$

$$\sum_{k=0}^{n}\binom{2n}{n-k}F_{3k}P_k=2^nF_n(6),\text{ where } P_k=F_k(2) \text{ is the } k^{\text{th}} \text{ Pell number;}\tag{4}$$

$$\sum_{k=0}^{n}\binom{2n}{n-k}F_k(x)F_k(x+1)=F_n(x^2+x+4);\tag{5}$$

$$\sum_{k=0}^{n}(-1)^{k+1}\binom{2n}{n-k}F_k(x)F_k(4/x)=\frac{1-(-1)^n}{2}\left(\frac{x^2+4}{x}\right)^{n-1},\ x\neq 0;\tag{6}$$

$$\sum_{k=0}^{n}\binom{2n}{n-k}F_k(x)^2=(x^2+4)^{n-1};\tag{7}$$

$$\sum_{k=0}^{n}(-1)^{k+1}\binom{2n}{n-k}F_k(x)^2=\frac{4^n-(-x^2)^n}{4+x^2};\tag{8}$$

$$\sum_{k=0}^{[\frac{n-1}{2}]}\binom{2n}{n-2k-1}F_{2k+1}(x)=x^{n-1}F_n(4/x).\tag{9}$$

The latter equation is the one given in H-500. **Hint:** Deduce (1) from the main identity of H-492.

H-519 ***Proposed by Paul S. Bruckman, Seattle, WA***

Let p denote a prime $\equiv 1 \pmod 4$.

(a) Prove that, for all $p \not\equiv 1 \pmod{24}$, there exist positive integers k, u, and v such that

(i) $k\,|\,u^2$;

(ii) $p+4k=(4u-1)(4v-1)$.

(b) Prove or disprove the conjecture that the restriction $p \not\equiv 1 \pmod{24}$ in part (a) may be removed, i.e., part (a) is true for all $p \equiv 1 \pmod 4$.

H-520 ***Proposed by Andrej Dujella, University of Zagreb, Croatia***

Let n be an integer. Prove that there exist an infinite set $D \subseteq \mathbf{N}$ with the property that for all $c, d \in D$ the integer $cd+n$ is not square free.

SOLUTIONS

Complex Situation

H-502 ***Proposed by Zdzislaw W. Trzaska, Warsaw, Poland***
(Vol. 33, no. 4, August 1995)

Given two sequences of polynomials in the complex variable $z \in C$ defined recursively as

$$T_k(z)=\sum_{m=0}^{k}a_{km}z^m,\quad k=0,1,2,\ldots,\tag{i}$$

with $T_0(z)=1$ and $T_1(z)=(1+z)T_0$, and

$$P_k(z)=\sum_{m=0}^{k}b_{km}z^m,\quad k=0,1,2,\ldots,\tag{ii}$$

with $P_0(z)=0$ and $P_1(z)=1$.

Prove that, for all $z \in C$ and $k=0,1,2,\ldots$, the equality

$$P_k(z)T_{k-1}(z)-T_k(z)P_{k-1}(z)=1\tag{iii}$$

holds.

Solution by the proposer

From (i), we have

$$a_{kp} = \left.\frac{\partial^p T_k(z)}{\partial z^p}\right|_{z=0}, \quad p = 0, 1, 2, \ldots, \tag{1}$$

so that we can write

$$T_2(z) = a_{20} + a_{21}z + a_{22}z^2 \tag{2}$$

with

$$a_{20} = 1, \quad a_{21} = 3, \quad a_{22} = 1. \tag{3}$$

Thus, the polynomial $T_k(z)$ fulfills the relation

$$T_{k+1}(z) = (2+z)T_k(z) - T_{k-1}(z), \quad k = 1, 2, \ldots. \tag{4}$$

Similarly, we can write

$$P_{k+1}(z) = (2+z)P_k(z) - P_{k-1}(z), \quad k = 0, 1, 2, \ldots, \tag{5}$$

with $P_0(z) = 0$ and $P_1(z) = 1$.

Note that coefficients of both polynomials belong to modified numerical triangles MNT1 and MNT2, respectively (see [1]).

Substituting the above results into LHS of (iii) gives

$$\begin{aligned} \text{LHS(iii)} &= P_k(z)T_{k+1}(z) - [(2+z)T_{k-1}(z) - T_{k-1}(z)]P_{k-1}(z) \\ &= [P_k(z) - (2+z)P_{k-1}(z)]T_{k-1}(z) + P_{k-1}(z)T_{k-2}(z). \end{aligned} \tag{6}$$

Next, using (2) yields

$$\text{LHS(iii)} = -P_{k-2}(z)T_{k-1}(z) + P_{k-1}(z)T_{k-2}(z). \tag{7}$$

Thus, repeating the above procedure $(k-1)$ times, we finally get

$$\text{LHS(iii)} = P_1(z)T_0(z) - T_1(z)P_0(z). \tag{8}$$

But, from (i) and (ii), we obtain

$$\text{LHS(iii)} = 1, \tag{9}$$

which means that LHS(iii) = RHS(iii), thus completing the proof.

Note that another proof can be presented by using the mathematical induction approach.

Reference

1. Z. Trzaska. "On Numerical Triangles Showing Links with Chebyshev Polynomials." *C. Lanczos Int. Cent. Conf.*, December 12-17, 1993, NCSU, Raleigh, NC.

A Complex Product

H-503 ***Proposed by Paul S. Bruckman, Edmonds, WA***
(Vol. 33, no. 5, November 1995)

Let $\mathcal{S}$ be the set of functions $F: C^3 \to C$ (C is the complex plane) satisfying the following formal properties:

$$xyz\,F(x, x^3y, x^3y^2z) = F(x, y, z); \tag{1}$$

$$F(x^{-1}, y, z^{-1}) = F(x, y, z). \tag{2}$$

Formally define the functions U and V as follows:

$$U(x,y,z)=\Sigma x^{n^3}y^{n^2}z^n \quad \text{(summed over all integers } n\text{);} \tag{3}$$

$$V(x,y,z)=\prod_{n=1}^{\infty}(1-y^{2n}A(x))(1+x^{3n^2-3n+1}y^{2n-1}z)(1+x^{-3n^2+3n-1}y^{2n-1}z^{-1}), \tag{4}$$

where

$$A(x)=\frac{\Sigma x^{3m}o_m}{\Sigma x^{3m}e_m} \quad \text{(summed over all integers } m\text{),} \tag{5}$$

$$o_m=\tfrac{1}{2}(1-(-1)^m), \qquad e_m=\tfrac{1}{2}(1+(-1)^m). \tag{6}$$

Show that, at least formally,

$$U\in\mathcal{S}, \qquad V\in\mathcal{S}; \tag{7}$$

$$A(1)=1; \tag{8}$$

$$U(1,y,z)=V(1,y,z). \tag{9}$$

Prove or disprove that $U(x,y,z)\equiv V(x,y,z)$ identically. Can $U(x,y,z)$ be factored into an infinite product?

Solution by the proposer

Proof that U satisfies (1): $xyzU(x,x^3y,x^3y^2z)=\Sigma x^{n^3+3n^2+3n+1}y^{n^2+2n+1}z^{n+1}=\Sigma x^{(n+1)^3}y^{(n+1)^2}z^{n+1}=\Sigma x^{n^3}y^{n^2}z^n=U(x,y,z)$.

Proof that U satisfies (2): $U(x^{-1},y,z^{-1})=\Sigma x^{-n^3}y^{n^2}z^{-n}=\Sigma x^{n^3}y^{(-n)^2}z^n=U(x,y,z)$. Therefore, $U\in\mathcal{S}$.

Proof that V satisfies (1): Let

$$P_n=P_n(x,y)=1-y^{2n}A(x),\ \ Q_n=Q_n(x,y,z)=1+x^{3n^2-3n+1}y^{2n-1}z,\ \ \overline{Q}_n=Q_n(x^{-1},y,z^{-1}).$$

Note that $x^{6n}A(x)=[\Sigma x^{3(m+2n)}o_m]\div[\Sigma x^{3m}e_m]=[\Sigma x^{3m}o_{m-2n}]\div[\Sigma x^{3m}e_m]=A(x)$; therefore, we have $P_n(x,x^3y)=1-y^{2n}x^{6n}A(x)=1-y^{2n}A(x)$, or

$$P_n(x,y)=P_n(x,x^3y). \tag{10}$$

Next,

$$\prod_{n=1}^{\infty}Q_n(x,x^3y,x^3y^2z)=\prod_{n=1}^{\infty}[1+x^{3n^2-3n+1+6n-3+3}y^{2n-1+2}z]$$
$$=\prod_{n=1}^{\infty}[1+x^{3n^2+3n+1}y^{2n+1}z]=\prod_{n=2}^{\infty}[1+x^{3n^2-3n+1}y^{2n-1}z],$$

or

$$\prod_{n=1}^{\infty}Q_n(x,x^3y,x^3y^2z)=(1+xyz)^{-1}\prod_{n=1}^{\infty}Q_n(x,y,z). \tag{11}$$

Also

$$\prod_{n=1}^{\infty}\overline{Q}_n(x,x^3y,x^3y^2z)=\prod_{n=1}^{\infty}[1+x^{-3n^2+3n-1+6n-3-3}y^{2n-1-2}z^{-1}]$$

$$=\prod_{n=1}^{\infty}[1+x^{-3n^2+9n-7}y^{2n-3}z^{-1}]=\prod_{n=0}^{\infty}[1+x^{-3n^2+3n-1}y^{2n-1}z^{-1}],$$

or

$$\prod_{n=1}^{\infty}\overline{Q}_n(x, x^3y, x^3y^2z)=(xyz)^{-1}(1+xyz)\prod_{n=1}^{\infty}\overline{Q}_n(x, y, z). \tag{12}$$

Combining (10), (11), and (12), we see that (at least formally),

$$\begin{aligned}V(x, x^3y, x^3y^2z)&=\prod_{n=1}^{\infty}P_n(x, x^3y)Q_n(x, x^3y, x^3y^2z)\overline{Q}_n(x, x^3y, x^3y^2z)\\&=(xyz)^{-1}\prod_{n=1}^{\infty}P_n(x, y)Q_n(x, y, z)\overline{Q}_n(x, y, z),\end{aligned}$$

or

$$xyzV(x, x^3y, x^3y^2z)=V(x, y, z). \tag{13}$$

Proof that V satisfies (2): Clearly $A(x^{-1})=A(x)$, so $P_n(x^{-1}, y)=P_n(x, y)$; also, $\overline{Q}_n(x^{-1}, y, z^{-1})=Q_n(x, y, z)$. It follows that $V(x^{-1}, y, z^{-1})=V(x, y, z)$. Therefore, $V\in\mathcal{S}$.

Proof of (8): Let $A_n(x)=\Sigma_{-n}^{n}x^{3m}o_m/\Sigma_{-n}^{n}x^{3m}e_m$. We readily find that $\Sigma_{-n}^{n}o_m=n+o_n$ and $\Sigma_{-n}^{n}e_m=n+e_n$. Thus, $A_n(1)=(n+o_n)/(n+e_n)$. Taking limits, we have $A(1)=\lim_{n\to\infty}A_n(1)=1$.

Proof of (9): Setting $x=1$ in (3) and (4) [using (8)], we find that $U(1, y, z)=\Sigma y^{n^2}z^n$ and $V(1, y, z)=\prod_{n=1}^{\infty}(1-y^{2n})(1+y^{2n-1}z)(1+y^{2n-1}z^{-1})$.

From this, we recognize that (9) is merely a statement of the famous triple-product identity of Jacobi. This is intimately connected with the theory of elliptic functions and, in particular, the Theta-functions studied by Jacobi.

Although the above results seem quite interesting and seem to imply some relationship between U and V, this relationship appears to be illusory, except for certain values of x. For one thing, the above results have only been demonstrated formally; a more rigorous treatment leads one to the conclusion that the series defining U and the product defining V are divergent unless $|x|=1$. Setting $x=\exp i\theta$, where θ is real, we may show that

$$A_n(x)=\left\{\frac{\sin 3(n+1)\theta}{\sin 3(n-1)\theta}\right\}^{(-1)^n}. \tag{14}$$

Otherwise, $A_n(x)$ has two distinct cluster points, namely, x^6 and x^{-6} (assuming $x\neq 0$), and therefore has no limit point, as $n\to\infty$. Even if $|x|=1$, however, the expression in (14) has no limit points, except for a finite set of values of x. In fact, it is not difficult to deduce the following result from (14):

$$\begin{aligned}&A(x)=(-1)^{[\frac{1}{4}(k+3)]-[\frac{1}{4}k]}, \text{ if } x=\exp(ki\pi/6), k \text{ integral;}\\&\text{otherwise, } A(x) \text{ is undefined.}\end{aligned} \tag{15}$$

Thus, $A(x)=1$ if $x^3=1$, in which case $P_n=1-y^{2n}$. Also, since $n^3\equiv n \pmod 6$, we see that, if $x^6=1$, $U(x, y, z)=U(1, y, xz)$ and $V(x, y, z)=V(1, y, xz)$. Jacobi's identity states that these last two quantities must be equal.

Experience suggests that if two expressions such as U and V are equal for certain special values of x (e.g., for which $|x|=1$), one should be able to employ analytic continuation to extend the equality for $|x| \neq 1$. However, as we have found, this extension is impossible; thus, $U \not\equiv V$ identically. Another way to show this is as follows: Since $(1+y/xz)$ is a factor of V, it follows that V has a zero (qua function of z) at $z=-y/x$. However, if we set $z=-y/x$ in the series defining U, this yields an expression that does *not* vanish identically, namely, $U(x,y,-y/x)= \Sigma(-1)^n x^{n^3-n} y^{n^2+n} = -y^2(1-x^{-6})+y^6(x^6-x^{-24})-y^{12}(x^{24}-x^{-60})+\cdots$; however, this *does* vanish at the special values for which $x^6=1$.

The factorization of U, if any such exists (and this seems doubtful), must be exotic indeed, and remains an open question. Toward this end, it would seem desirable, if possible, to replace $A_n(x)$ by some other function that is better behaved, while still satisfying the appropriate criteria. Also, the functions Q_n and $\overline{Q}_n$ might, conceivably, by replaced by other, more esoteric expressions that still satisfy the desired conditions of the factorization problem. The general problem may be stated in the following way. Given $U(x,y,z)$ as defined by (3), find a factorization as follows:

$$U(x,y,z) \equiv \prod_{n=1}^{\infty} S_n(x,y,z) \quad \text{(valid for appropriate convergence criteria),} \tag{16}$$

where the S_n's satisfy the conditions:

$$S_n(x^{-1},y,z^{-1}) = S_n(x,y,z); \tag{17}$$

$$xyz\prod_{n=1}^{\infty} S_n(x,x^3y,x^3y^2z) = \prod_{n=1}^{\infty} S_n(x,y,z); \tag{18}$$

$$S_n(1,y,z) = (1-y^{2n})(1+y^{2n-1}z)(1+y^{2n-1}z^{-1}). \tag{19}$$

The proposer of this research problem is indebted to his former mentor, A. O. L. Atkin at the University of Illinois in Chicago, for providing helpful hints and suggestions. Moreover, Dr. Atkin suggested another area of possible extension, namely, to work with the sum $\Sigma w^{n^4} x^{n^3} y^{n^2} z^n$; this latter sum has fewer convergence problems than the sum proposed in this problem but it has the undesirable quality of being more complicated. This is as far as this proposer took this problem. All comments are invited from the readers.

Also solved by A. Dujella.

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