ON A KIND OF GENERALIZED ARITHMETIC-GEOMETRIC PROGRESSION

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1. INTRODUCTION

Let $(t|h)_p$ denote the generalized falling factorial of t of degree p and increment h, namely, $(t|h)_0 = 1$ and

$$(t|h)_p = \prod_{j=0}^{p-1}(t-jh), \quad p = 1, 2, 3, \dots.$$

In particular, $(t|1)_p$ is denoted by $(t)_p$. The main purpose of this note is to establish an explicit summation formula for a kind of generalized arithmetic-geometric progression of the form

$$S_{a,h,p}(n) = \sum_{k=1}^{n} a^k (k|h)_p,$$

where a and h are real or complex numbers, and p a positive integer. It is always assumed that $a \neq 0$ and $a \neq 1$.

It is known that the sum $S_{a,0,p}(n) = \sum_{k=1}^{n} a^k k^p$ has been investigated with different methods by de Bruyn [1] and Gauthier [6]. De Bruyn developed some explicit formulas by using certain determinant expressions derived from Cramer's rule, and Gauthier made repeated use of the differential operator $D = x(d/dx)$ to express $S_{a,0,p}(n)$ as a^n times a polynomial of degree p in n, plus an n-independent term in which the coefficients are determined recursively. In this note we shall express the general sum $S_{a,h,p}(n)$ explicitly in terms of the degenerate Stirling numbers due to Carlitz [2]. In particular, an explicit formula for $S_{a,0,p}(n)$ will be given via Stirling numbers of the second kind. Finally, as other applications of our Lemma 1, some combinatorial sums involving generalized factorials will be presented.

2. AN EXPLICIT SUMMATION FORMULA

We will make use of the degenerate Stirling numbers $S(n, k|\lambda)$ first defined by Carlitz [2] using the generating function

$$\frac{1}{k!}((1+\lambda x)^{1/\lambda} - 1)^k = \sum_{n=k}^{\infty} S(n, k|\lambda)\frac{x^n}{n!}. \tag{1}$$

The ordinary Stirling numbers of the second kind are given by

$$S(n, k) := S(n, k|0) = \lim_{\lambda \to 0} S(n, k|\lambda).$$

Also, Carlitz proved among other things (cf. [2])

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$$(t|h)_p = \sum_{j=0}^{p} S(p, j|h)(t)_j \tag{2}$$

$$S(p, j|h) = \frac{1}{j!}\sum_{k=0}^{j}(-1)^{j-k}\binom{j}{k}(k|h)_p. \tag{3}$$

Note that the so-called C-numbers extensively studied by Charalambides and others (see [3] and [4]) are actually related to the degenerated Stirling numbers in such a way that

$$S(p, j|h) = h^p C(p, j, 1/h), \ (h \neq 0).$$

The following lemma may be regarded as a supplement to the simple summation rule proposed in [7].

Lemma 1: Let $F(n, k)$ be a bivariate function defined for integers $n, k \geq 0$. If there can be found a formula such as

$$\sum_{k=0}^{n} F(n, k)\binom{k}{j} = \psi(n, j), \ (j \geq 0), \tag{4}$$

then for every integer $p \geq 0$ we have a summation formula

$$\sum_{k=0}^{n} F(n, k)(k|h)_p = \sum_{j=0}^{p} \psi(n, j) j! S(p, j|h) \tag{5}$$

with its limiting case for $h \to 0$,

$$\sum_{k=0}^{n} F(n, k)k^p = \sum_{j=0}^{p} \psi(n, j) j! S(p, j). \tag{6}$$

Proof: Plainly, (5) may be verified at once by substituting expression (2) with $t = k$ into the left-hand side of (5) and by changing the order of summation and using (4), namely,

$$\sum_{k=0}^{n} F(n, k)\sum_{j=0}^{p} S(p, j|h) j!\binom{k}{j} = \sum_{j=0}^{p} S(p, j|h) j! \psi(n, j).$$

The fact that (5) implies (6) with $h \to 0$ is also evident. □

Lemma 2: For $a \neq 0$ and $a \neq 1$, let $\phi(n, j) \equiv \phi(n, j; a)$ be defined by

$$\phi(n, j) = \sum_{k=0}^{n} a^k \binom{k}{j}, \ (j \geq 0). \tag{7}$$

Then $\phi(n, j)$ satisfies the recurrence relations

$$a\phi(n, j-1) + (a-1)\phi(n, j) = a^{n+1}\binom{n+1}{j} \tag{8}$$

with $\phi(n, 0) = (a^{n+1} - 1)/(a-1)$ and $j = 1, 2, \ldots$.

Proof: Evidently, we have

$$\begin{aligned}
a^{n+1}\binom{n+1}{j} &= \phi(n+1, j) - \phi(n, j) \quad (j \ge 1) \\
&= \sum_{k=0}^{n} a^{k+1}\binom{k+1}{j} - \phi(n, j) \\
&= a\sum_{k=0}^{n} a^k\left[\binom{k}{j} + \binom{k}{j-1}\right] - \sum_{k=0}^{n} a^k\binom{k}{j} \\
&= (a-1)\phi(n, j) + a\phi(n, j-1),
\end{aligned}$$

where the initial condition is given by $\phi(n, 0) = \sum_{k=0}^{n} a^k = (a^{n+1}-1)/(a-1)$. □

In what follows we will occasionally make use of the forward difference operator Δ, defined by $\Delta f(x) = f(x+1) - f(x)$ and $\Delta^j = \Delta\Delta^{j-1}, (j \ge 2)$.

Proposition: The following summation formulas hold:

$$S_{a,h,p}(n) = \sum_{j=1}^{p} \phi(n, j; a) j!\, S(p, j \mid h); \tag{9}$$

$$S_{a,0,p}(n) = \sum_{j=1}^{p} \phi(n, j; a) j!\, S(p, j). \tag{10}$$

These formulas may also be written as

$$S_{a,h,p}(n) = \sum_{j=1}^{p} \phi(n, j; a)\left[\Delta^j (t \mid h)_p\right]_{t=0} \tag{11}$$

and

$$S_{a,0,p}(n) = \sum_{j=1}^{p} \phi(n, j; a)\left[\Delta^j t^p\right]_{t=0}, \tag{12}$$

where $\phi(n, j; a)$ $(1 \le h \le p)$ and the higher differences involved have explicit expressions, viz.,

$$\phi(n, j; a) = \frac{1}{1-a}\left[\left(\frac{a}{1-a}\right)^j - a^{n+1}\sum_{r=0}^{j}\binom{n+1}{j-r}\left(\frac{a}{1-a}\right)^r\right], \tag{13}$$

$$\left[\Delta^j (t \mid h)_p\right]_{t=0} = \sum_{r=0}^{j}(-1)^{j-r}\binom{j}{r}(r \mid h)_p, \tag{14}$$

$$\left[\Delta^j t^p\right]_{t=0} = \Delta^j 0^p = \sum_{r=0}^{j}(-1)^{j-r}\binom{j}{r} r^p. \tag{15}$$

Proof: Comparing (7) with (4), in which $F(n, k)$ corresponds to a^k (not depending on n), we see that (9) and (10) are merely consequences of Lemma 1. Thus, it suffices to verify (11) and (13). Actually (14) and (15) are well-known expressions from the calculus of finite differences. Consequently, the equivalence between (9) and (11) and that between (10) and (12) both follow from the relations (3) and (14).

Moreover, Lemma 2 implies the following:

$$\begin{aligned}\phi(n, j) &= \frac{a^{n+1}}{a-1}\sum_{r=0}^{j-1}\binom{n+1}{j-r}\left(\frac{a}{a-1}\right)^r + \left(\frac{a}{1-a}\right)^j \phi(n, 0)\\ &= \frac{a^{n+1}}{a-1}\sum_{r=0}^{j}\binom{n+1}{j-r}\left(\frac{a}{1-a}\right)^r + \frac{1}{1-a}\left(\frac{a}{1-a}\right)^j.\end{aligned}$$

This is precisely equivalent to (13). □

Remark: According to the terminology adopted in Comtet's book [5], we may say that both (11) and (12) provide summation formulas of rank 3 as they both consist of triple sums after having substituted (13), (14), and (15) into the right-hand sides of (11) and (12), respectively. The number of terms involved in each formula is, obviously,

$$\sum_{j=1}^{p}(j+2)(j+1) = \frac{1}{3}(p+3)(p+2)(p+1) - 2.$$

Surely these formulas are of practical value for computation when $n \gg p$.

Let us rewrite (10) in the form

$$\sum_{k=1}^{n} a^k k^p = \sum_{j=1}^{p} \frac{j!S(p, j)}{1-a}\left[\left(\frac{a}{1-a}\right)^j - a^{n+1}\sum_{r=0}^{j}\binom{n+1}{j-r}\left(\frac{a}{1-a}\right)^r\right], \tag{10'}$$

where $j!S(p, j) = \Delta^j 0^p$ are given by (15). Notice that for $|a| < 1$ the limit of $a^{n+1}\binom{n+1}{r}$ $(0 \le r \le j)$ is zero when $n \to \infty$. Thus, using (10′) and passing to limit, we easily obtain the following convergent series:

$$\sum_{k=1}^{\infty} a^k k^p = \sum_{j=1}^{p} j!S(p, j)\frac{a^j}{(1-a)^{j+1}}, \quad (|a| < 1).$$

3. EXAMPLES

Example 1: Let the Fibonacci numbers F_k and the Lucas numbers L_k be given by the Binet forms

$$F_k = (\alpha^k - \beta^k)/\sqrt{5} \quad \text{and} \quad L_k = \alpha^k + \beta^k$$

with $\alpha = (a+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$. Then it is easily seen that the following sums, i.e.,

$$\Phi(n) = \sum_{k=1}^{n}(k|h)_p F_k \quad \text{and} \quad \Lambda(n) = \sum_{k=1}^{n}(k|h)_p L_k,$$

can be computed by means of (9) or (11). Indeed, we have

$$\Phi(n) = \frac{1}{\sqrt{5}}\sum_{j=1}^{p}(\phi(n, j; \alpha) - \phi(n, j; \beta))j!S(p, j|h),$$

and

$$\Lambda(n) = \sum_{j=1}^{p}(\phi(n, j; \alpha) + \phi(n, j; \beta))j!S(p, j|h),$$

where $\phi(n, j; \alpha)$ and $\phi(n, j; \beta)$ are given by (13) with $a = \alpha$ and $a = \beta$.

Example 2: Given real numbers h and θ with $0<\theta<2\pi$. It is easily found that the sums

$$C(n)=\sum_{k=1}^{n}(k\,|\,h)_p\cos k\theta \quad\text{and}\quad S(n)=\sum_{k=1}^{n}(k\,|\,h)_p\sin k\theta$$

can also be computed via (9) or (11). To see this, let us take $a=e^{i\theta}$ with $i=\sqrt{-1}$. Evidently, the sums $C(n)$ and $S(n)$ are given by the real and imaginary parts of the sum formula for $S_{a,h,p}(n)$, namely,

$$C(n)=\operatorname{Re}\sum_{j=1}^{p}\phi(n,j;e^{i\theta})j!S(p,j\,|\,h) \quad\text{and}\quad S(n)=\operatorname{Im}\sum_{j=1}^{p}\phi(n,j;e^{i\theta})j!S(p,j\,|\,h).$$

Example 3: Let $\phi(n,j)\equiv\phi(n,j;a)$ be given by (13). Then, using the values of $S(p,j)$ for $j\le p\le 5$, we can immediately write down several special formulas for the sum $S_{a,p}(n)=\sum_{k=1}^{n}a^kk^p$, as follows:

$$\begin{aligned}
&S_{a,1}(n)=\phi(n,1),\\
&S_{a,2}(n)=\phi(n,1)+2\phi(n,2),\\
&S_{a,3}(n)=\phi(n,1)+6\phi(n,2)+6\phi(n,3),\\
&S_{a,4}(n)=\phi(n,1)+14\phi(n,2)+36\phi(n,3)+24\phi(n,4),\\
&S_{a,5}(n)=\phi(n,1)+30\phi(n,2)+150\phi(n,3)+240\phi(n,4)+120\phi(n,5).
\end{aligned}$$

As may be verified, the first two equalities given above do agree with the two explicit expressions displayed in de Bruyn [1].

4. OTHER APPLICATIONS OF LEMMA 1

It is clear that the key step necessary for applying the summation rule given by Lemma 1 is to find an available form of $\psi(n,j)$ with respect to a given $F(n,k)$. Now let us take $F(n,k)$ to be the following forms, respectively,

$$1,\ \binom{n}{k},\ \binom{n}{k}^2,\ \binom{n}{2k},\ \binom{n}{2k+1},\ \binom{s+k}{s},\ H_k,$$

where $H_k=1+1/2+\cdots+1/k$ are harmonic numbers. Then the corresponding $\psi(n,j)$'s may be found easily by using some known combinatorial identities, and, consequently, we obtain the following sums (with $p\ge 1$) via (5):

$$\sum_{k=1}^{n}(k\,|\,h)_p=\sum_{j=1}^{p}\binom{n+1}{j+1}j!S(p,j\,|\,h), \tag{16}$$

$$\sum_{k=1}^{n}\binom{n}{k}(k\,|\,h)_p=\sum_{j=1}^{p}\binom{n}{j}2^{n-j}j!S(p,j\,|\,h), \tag{17}$$

$$\sum_{k=1}^{n}\binom{n}{k}^2(k\,|\,h)_p=\sum_{j=1}^{p}\binom{2n-j}{n}(n)_jS(p,j\,|\,h), \tag{18}$$

$$\sum_{k=1}^{[n/2]}\binom{n}{2k}(k\,|\,h)_p=\sum_{j=1}^{p}2^{n-2j}\binom{n-j}{j}\frac{n}{n-j}j!S(p,j\,|\,h), \tag{19}$$

$$\sum_{k=1}^{[n/2]} \binom{n}{2k+1} (k|h)_p = \sum_{j=1}^{p} 2^{n-2j} \binom{n-j}{j} j!S(p, j|h), \tag{20}$$

$$\sum_{k=1}^{n} \binom{s+k}{s} (k|h)_p = \sum_{j=1}^{p} \binom{n+1}{j} \binom{n+1+s}{s} \frac{n+1-j}{s+1+j} j!S(p, j|h), \tag{21}$$

$$\sum_{k=1}^{n} (k|h)_p H_k = \sum_{j=1}^{p} \binom{n+1}{j+1} \left(H_{n+1} - \frac{1}{j+1} \right) j!S(p, j|h). \tag{22}$$

These sums with $h=0$ will reduce to the cases displayed in [7]. Actually, (19) and (20) follow from an application of the pair of Moriarty identities, (21) from that of Knuth's identity, and (22) is obviously implied by (4)-(5) and the well-known relation

$$\sum_{k=j}^{n} \binom{k}{j} H_k = \binom{n+1}{j+1} \left(H_{n+1} - \frac{1}{j+1} \right) \quad \text{(see §2 of [7]).}$$

Evidently, (21) implies the classical formula for $\sum_{k=1}^{n} k^p$ when $s=0$ and $h=0$.

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