ON MULTIPLICITY SEQUENCES

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The concept of divisibility sequence is quite popular in the mathematical literature. Starting from [1], where Marshall Hall called a sequence g of rational integers a **divisibility sequence** iff

$$\forall_{m,n\in N} \quad m|n \Rightarrow g(m)|g(n), \qquad \textbf{(DS)}$$

numerous papers appeared (see, e.g., [6], [7]). Another study of such sequences was initiated by Kimberling who in [2] called g a **strong divisibility sequence** iff

$$\forall_{m,n\in N} \quad \text{G.C.D.}(g(m), g(n)) = g(\text{G.C.D.}(m, n)). \qquad \textbf{(SDS)}$$

It is obvious that $\mathbf{SDS} \Rightarrow \mathbf{DS}$. If we take a sequence g defined by $g(2^k(2m+1)) = 2^{k(2m+1)}$, we get a **DS** sequence which is not a **SDS** sequence.

The problem of characterizing polynomial **DS** sequences was taken up in [3] and [4]. It was proved in [4] that polynomial **DS** sequences are exactly those of the form $g(n) = an^k$.

As the concept of LCM of rational integers is "parallel" with the GCD of rational integers, it is natural to introduce the following definition: g is a **multiplicity sequence** iff

$$\forall_{m,n\in N} \quad \text{L.C.M.}(g(m), g(n)) = g(\text{L.C.M.}(m, n)). \qquad \textbf{(MS)}$$

The sequence of the Fibonacci numbers is a **SDS** sequence but not a **MS** sequence. Another example of **SDS** not **MS** sequence is $g(n) = 2^n - 1$.

Theorem: $\mathbf{MS} \Rightarrow \mathbf{SDS}$.

Proof:

First step. We shall assume that g is multiplicative $(\text{G.C.D.}(m, n) = 1 \Rightarrow g(mn) = g(m)g(n))$. In this case, we actually have $\mathbf{MS} \Leftrightarrow \mathbf{SDS}$. In fact, let us note that for the multiplicative sequence g we have

$$g(m)g(n) = g(\text{G.C.D.}(m, n))g(\text{L.C.M.}(m, n)) \qquad (1)$$

for any $m, n \in N$. So, if g is **MS**, then by (1) we get

$$g(\text{G.C.D.}(m, n)) = \frac{g(m)g(n)}{g(\text{L.C.M.}(m, n))} = \frac{g(m)g(n)}{\text{L.C.M.}(g(m), g(n))} = \text{G.C.D.}(g(m), g(n)).$$

Analogously, we can show that $\mathbf{SDS} \Rightarrow \mathbf{MS}$.

Second step. Suppose g is a **MS** sequence. Thus,

$$g(m)|\text{L.C.M.}(g(m), g(n)) = g(\text{L.C.M.}(m, n)),$$
$$g(n)|\text{L.C.M.}(g(m), g(n)) = g(\text{L.C.M.}(m, n)),$$

and $g(m)g(n) = cg(\text{L.C.M.}(m, n))$. Therefore, if $\text{G.C.D.}(m, n) = 1$, then

$$g(m)g(n) = cg(mn). \qquad (2)$$

Sequences (functions) that satisfy (2) are called **quasi-multiplicative** (see [5]). We note that $c = g(1)$ and $G(n) = \frac{g(n)}{g(1)}$ is **MS**, which is also multiplicative. Hence,

$$\begin{aligned} \text{G.C.D.}(g(m), g(n)) &= \text{G.C.D.}(g(1)G(m), g(1)G(n)) \\ &= g(1)\text{G.C.D.}(G(m), G(n)) \\ &= g(1)G(\text{G.C.D.}(m, n)) \\ &= g(\text{G.C.D.}(m, n)). \end{aligned}$$

Remark: It follows from the Theorem and from Monzingo's result [4] that if $g(n)$ is a polynomial **MS** sequence, then $g(n) = an^k$.

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When Richard Vine retired as an administrator at Lockheed Corporation some years ago, The Fibonacci Association was the lucky winner because Richard brought all of his very able talents to his job as Subscription Manager of *The Fibonacci Quarterly*. Richard also belonged to a local tennis club where he was active on the court as well as with administrative duties. Furthermore, Richard had an extremely beautiful voice and sang as a professional actor in such plays as "Paint Your Wagon." Frequently, when conversing with Richard over the phone or while he was visiting with a local member of the Board of Directors concerning a Fibonacci chore, he would tell the story of the week from the tennis club. To wit: What do you get when you cross a pitbull with a collie?...A dog that bites you and then goes for help.

After 17 years of taking subscription and book orders with an extra bit of special care and flair, Richard Vine has decided to retire as our Subscription Manager. Richard, the members of the Board of Directors of the Fibonacci Association and the Editor of *The Fibonacci Quarterly*, who never could have done his job so well without your help, want to offer you a big **thank you** for a job splendidly done. You shall definitely be missed.