ON A QUESTION OF COOPER AND KENNEDY

Helmut Prodinger
Department for Algebra and Discrete Mathematics
TU Wien, Wiedner Hauptstrasse 8-10/118, A-1040 Wien, Austria
email: Helmut.Prodinger@tuwien.ac.at
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In [2], Cooper and Kennedy note the following: If

$$x_n = ax_{n-1} + bx_{n-2} + cx_{n-3}, \tag{1}$$

then

$$x_n^2 = Ax_{n-1}^2 + Bx_{n-2}^2 + Cx_{n-3}^2 + Dx_{n-4}^2 + Ex_{n-5}^2 + Fx_{n-6}^2, \tag{2}$$

where the coefficients A, B, C, D, E, F may be expressed in terms of a, b, c. They ask: **Is there a similar formula for third powers**? The answer is: **YES**. The reason is the following: Sequences which are solutions of linear recurrences with constant coefficients have ordinary generating functions which are *rational*. Conversely, if a rational function has no pole in $z = 0$, its Taylor coefficients fulfill a linear recurrence with constant coefficients. If

$$f := \sum_{n\ge 0} a_n z^n \quad \text{and} \quad g := \sum_{n\ge 0} b_n z^n \tag{3}$$

are two (formal) series, their HADAMARD product is defined to be

$$f \odot g := \sum_{n\ge 0} a_n b_n z^n. \tag{4}$$

And rational functions are closed under the Hadamard product! (See [1], p. 85.)

The larger (any maybe even more important) class of *holonomic* functions (solutions of linear differential equations with polynomial coefficients) is also closed under the Hadamard product. Their Taylor coefficients fulfill linear recursions with polynomial coefficients. There is a MAPLE package, GFUN, which computes (among many other things) the Hadamard product (see [4]).

There is another very useful program, EKHAD, written by Doron Zeilberger [3], which should be mentioned. With it, we find, for example, recursions for the d^{th} powers of the Fibonacci numbers F_n in almost no time. In the following, F_n^d will be a solution of the given recursion.

$d=1 \qquad x_{n+2} - x_{n+1} - x_n = 0,$

$d=2 \qquad x_{n+3} - 2x_{n+2} - 2x_{n+1} + x_n = 0,$

$d=3 \qquad x_{n+4} - 3x_{n+3} - 6x_{n+2} + 3x_{n+1} + x_n = 0,$

$d=4 \qquad x_{n+5} - 5x_{n+4} - 15x_{n+3} + 15x_{n+2} + 5x_{n+1} - x_n = 0,$

$d=5 \qquad x_{n+6} - 8x_{n+5} - 40x_{n+4} + 60x_{n+3} + 40x_{n+2} - 8x_{n+1} - x_n = 0,$

$d=6 \qquad x_{n+7} - 13x_{n+6} - 104x_{n+5} + 260x_{n+4} + 260x_{n+3} - 104x_{n+2} - 13x_{n+1} + x_n = 0.$

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3. M. Petkovsek, H. Wilf, & D. Zeilberger. $A = B$. A. K. Peters, Ltd, 1996.
4. Bruno Salvy & Paul Zimmermann. "GFUN: A Maple Package for the Manipulation of Generating and Holonomic Functions in One Variable." *ACM Transactions on Mathematical Software* **20.2** (1994):163-67.

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