ADVANCED PROBLEMS AND SOLUTIONS

Edited by
Raymond E. Whitney

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-529 ***Proposed by Paul S. Bruckman, Highwood, IL***

Let ρ denote the set of Pythagorean triples (a, b, c) such that $a^2+b^2=c^2$. Find all pairs of integers $m, n>0$ such that $(a, b, c)=(F_mF_n, F_{m+1}F_{n+2}, F_{m+2}F_{n+1})\in\rho$.

H-530 ***Proposed by Andrej Dujella, University of Zagreb, Croatia***

Let $k(n)$ be the period of a sequence of Fibonacci numbers $\{F_i\}$ modulo n. Prove that $k(n)\leq 6n$ for any positive integer n. Find all positive integers n such that $k(n)=6n$.

H-531 ***Proposed by Paul S. Bruckman, Highwood, IL***

Consider the sum $S=\sum_{n=1}^{\infty}t(n)/n^2$, where $t(1)=1$ and $t(n)=\prod_{p|n}(1-p^{-2})^{-1}$, $n>1$, the product taken over all prime p dividing n. Evaluate S and show that it is rational.

SOLUTIONS

Comment by H.-J. Seiffert

Correction: The identity of Problem H-510 should read

$$P_n=\sum_{k\in A_n}(-1)^{[(3k-2n+3)/4]}\,2^{[3k/2]}\binom{n+k}{2k+1}.$$

The proposer's solution, however, is correct. The mistake arose in the very last step, when replacing n by $n-1$. Indeed, H-510 is the proposer's first (incorrect) version of H-476.

Continued

H-509 ***Proposed by Paul S. Bruckman, Salmiya, Kuwait (Vol. 34, no. 2, May 1996)***

The *continued fractions* (*base k*) are defined as follows:

$$[u_1, u_2, \dots, u_n]_k=u_1+\frac{k}{u_{2^+}}\frac{k}{u_{3^+}}\cdots\frac{k}{u_n},\quad n=1,2,\dots, \tag{1}$$

where k is an integer $\neq 0$ and $(u_i)_{i=1}^{\infty}$ is an arbitrary sequence of real numbers.

Given a prime p with $(\frac{-k}{p}) = 1$ (Legendre symbol) and $k \not\equiv 0 \pmod p$, let h be the solution of the congruence

$$h^2 \equiv -k \pmod p, \text{ with } 0 < h < \tfrac{1}{2}p. \tag{2}$$

Suppose a symmetric continued fraction (base k) exists, such that

$$\frac{p}{h} = [a_1, a_2, \ldots, a_{n+1}, a_{n+1}, \ldots, a_1]_k, \tag{3}$$

where the a_i's are integers, n is even, and $k \mid a_i$, $i = 2, 4, \ldots, n$. Then show that integers x and y exist, with $\text{g.c.d.}(x, y) = 1$, given by

$$\frac{x}{y} = [a_{n+1}, \ldots, a_1]_k \tag{4}$$

that satisfy

$$x^2 + ky^2 = p. \tag{5}$$

Solution by the proposer

Let $[u_1, u_2, \ldots, u_n]_k = p_n / q_n$, $n = 1, 2, \ldots,$ define the n^{th} convergent of the c.f. (base k), assuming that the u_i's are integers. The p_n's and q_n's satisfy the common recurrence

$$z_n = u_n z_{n-1} + k z_{n-2}, \; n = 3, 4, \ldots. \tag{6}$$

Also, $p_1 / q_1 = [u_1]_k = u_1 / 1$ and $p_2 / q_2 = [u_1, u_2]_k = u_1 + k / u_2 = (u_1 u_2 + k) / u_2$, which yields the initial conditions

$$p_1 = u_1, \; q_1 = 1; \; p_2 = u_1 u_2 + k, \; q_2 = u_2. \tag{7}$$

First, we need some results concerning c.f.'s (base k), which we state as lemmas and prove by induction.

Lemma 1: Let p_n / q_n and p_{n+1} / q_{n+1} denote successive convergents of a c.f. (base k). Let $w_n = p_n q_{n+1} - p_{n+1} q_n$, $n = 1, 2, \ldots.$ Then

$$w_n = (-k)^n. \tag{8}$$

Proof: Let S_1 denote the set of positive integers n satisfying (8). Now $w_1 = u_1 \cdot u_2 - (u_1 u_2 + k) \cdot 1 = -k = (-k)^1$; hence, $1 \in S_1$.

Suppose $n \in S_1$. Then we get $w_{n+1} = p_{n+1} q_{n+2} - p_{n+2} q_{n+1} = p_{n+1}(u_{n+2} q_{n+1} + k q_n) - (u_{n+2} p_{n+1} + k p_n) q_{n+1} = -k(p_n q_{n+1} - p_{n+1} q_n) = -k w_n = -k(-k)^n$ (by the inductive hypothesis), or $w_{n+1} = (-k)^{n+1}$. Thus, $n \in S_1 \Rightarrow (n+1) \in S_1$. The result follows by induction. □

Lemma 2: Let $p_n / q_n = [u_1, u_2, \ldots, u_n]_k$, where the u_i's are integers with $k \mid u_i$, $i = 2, 4, 6, \ldots,$ for $n = 1, 2, \ldots.$ Furthermore, suppose the p_n's and q_n's are the integers naturally produced in the c.f. (base k) expansion, applying the recurrence relation in (6) and the initial conditions in (7). Then, for all even n,

$$\text{g.c.d.}(p_{n-1}, q_{n-1}) = |k|^{\frac{1}{2}n-1}, \tag{9}$$

$$\text{g.c.d.}(p_n, q_n) = |k|^{\frac{1}{2}n}. \tag{10}$$

Proof: Let S_2 denote the set of even positive integers n for which (9) and (10) are valid. Clearly, $\text{g.c.d.}(p_1, q_1) = 1$, since $q_1 = 1$. Note that $1 = |k|^{2/2-1}$. Also, since $k \mid u_2$, it follows that

$k|(u_1u_2+k)$. Thus, $1\cdot(u_1u_2/k+1)-u_1u_2/k=1$, which implies g.c.d.$(p_2/k, q_2/k)=1$; hence, g.c.d.$(p_2,q_2)=|k|=|k|^{2/2}$. We thus see that $2\in S_2$.

Suppose $n\in S_2$ and $p_{n-1}=(-k)^{\frac{1}{2}n-1}p'_{n-1}, q_{n-1}=(-k)^{\frac{1}{2}n-1}q'_{n-1}$, $p_n=(-k)^{\frac{1}{2}n}p'_n$, $q_n=(-k)^{\frac{1}{2}n}q'_n$, where g.c.d.$(p'_{n-1},q'_{n-1})=$ g.c.d.$(p'_n,q'_n)=1$. Then we have $p_{n+1}=u_{n+1}p_n+kp_{n-1}=(-k)^{\frac{1}{2}n}p'_{n+1}$, where $p'_{n+1}=u_{n+1}p'_n-p'_{n-1}$; similarly, $q_{n+1}=(-k)^{\frac{1}{2}n}q'_{n+1}$, where $q'_{n+1}=u_{n+1}q'_n-q'_{n-1}$. Therefore, $p_nq_{n+1}-p_{n+1}q_n=(-k)^n(p'_nq'_{n+1}-p'_{n+1}q'_n)=(-k)^n$ (using Lemma 1), so $p'_nq'_{n+1}-p'_{n+1}q'_n=1$. Then g.c.d.$(p'_{n+1},q'_{n+1})=1$, which implies g.c.d.$(p_{n+1},q_{n+1})=|k|^{\frac{1}{2}n}=|k|^{\frac{1}{2}(n+2)-1}$. This is the statement of (9) for $(n+2)$.

Again supposing $n\in S_2$, let $u_{n+2}=-ku'_{n+2}$ (since $k|u_{n+2}$). Then we get $p_{n+2}=u_{n+2}p_{n+1}+kp_n=$ $(-k)(-k)^{\frac{1}{2}n}u'_{n+2}p'_{n+1}-(-k)(-k)^{\frac{1}{2}n}p'_n=(-k)^{1+\frac{1}{2}n}p'_{n+2}$, where $p'_{n+2}=u'_{n+2}p'_{n+1}-p'_n$; similarly, $q_{n+2}=$ $(-k)^{1+\frac{1}{2}n}q'_{n+2}$, where $q'_{n+2}=u'_{n+2}q'_{n+1}-q'_n$. Then $p_{n+1}q_{n+2}-p_{n+2}q_{n+1}=(-k)^{\frac{1}{2}n}(-k)^{1+\frac{1}{2}n}(p'_{n+1}q'_{n+2}-$ $p'_{n+2}q'_{n+1})=(-k)^{n+1}$ (using Lemma 1), so $p'_{n+1}q'_{n+2}-p'_{n+2}q'_{n+1}=1$. Therefore, g.c.d.$(p'_{n+2},q'_{n+2})=1$, which implies g.c.d.$(p_{n+2},q_{n+2})=|k|^{1+\frac{1}{2}n}=|k|^{\frac{1}{2}(n+2)}$. This is the statement of (10) for $(n+2)$. Thus, $n\in S_2 \Rightarrow (n+2)\in S_2$. Since $2\in S_2$, the results follow by induction. □

Lemma 3: If $p_n/q_n=[u_1,u_2,\dots,u_n]_k$, $n=1,2,\dots$, then

$$[u_n,u_{n-1},\dots,u_2]_k=q_n/q_{n-1} \text{ and } [u_n,u_{n-1},\dots,u_1]_k=p_n/p_{n-1},\ n=2,3,\dots. \tag{11}$$

Proof: Let S_3 denote the integers $n\ge 2$ for which (11) is valid. Note that $[u_2]_k=u_2/1=$ q_2/q_1 and $[u_2,u_1]_k=u_2+k/u_1=(u_1u_2+k)/u_1=p_2/p_1$ [using (7)]. Therefore, $2\in S_2$.

Suppose $n\in S_3$. Then we get $[u_{n+1},u_n,\dots,u_2]_k=u_{n+1}+k/[u_n,\dots,u_2]_k=u_{n+1}+k/(q_n/q_{n-1})=$ $(u_{n+1}q_n+kq_{n-1})/q_n=q_{n+1}/q_n$ [using (6)]. Also $[u_{n+1},u_n,\dots,u_1]_k=u_{n+1}+k/[u_n,\dots,u_1]_k=u_{n+1}+$ $k/(p_n/p_{n-1})=(u_{n+1}p_n+kp_{n-1})/p_n=p_{n+1}/p_n$. Thus, $n\in S_3\Rightarrow(n+1)\in S_3$. Since $2\in S_3$, the result follows by induction. □

Also, we will make use of the following identity:

$$(a^2+kb^2)(c^2+kd^2)=(ac+kbd)^2+k(ad-bc)^2. \tag{12}$$

Now suppose $p_i/q_i=[a_1,a_2,\dots,a_i]_k$, $i=1,2,\dots,n+1$, in the sense described in the hypothesis of Lemma 2. Then $p_n=(-k)^{\frac{1}{2}n}p'_n$, $q_n=(-k)^{\frac{1}{2}n}q'_n$, $p_{n+1}=(-k)^{\frac{1}{2}n}p'_{n+1}$, and $q_{n+1}=(-k)^{\frac{1}{2}n}q'_{n+1}$, where g.c.d.$(p'_n,q'_n)=$ g.c.d.$(p'_{n+1},q'_{n+1})=1$. Moreover, $p'_nq'_{n+1}-p'_{n+1}q'_n=1$. Also, using Lemma 3, $[a_{n+1},\dots,a_2]_k=q_{n+1}/q_n$ and $[a_{n+1},\dots,a_1]_k=p_{n+1}/p_n$. The n^{th} and $(n+1)^{\text{st}}$ convergents of the c.f. (base k) given by (3) are p_n/q_n and p_{n+1}/q_{n+1}, respectively; the "remainder" of this c.f. is equal to p_{n+1}/p_n, which assumes the role of u_{n+2}. Thus, the value of the c.f. (base k) in (3) is given by

$$\frac{(p_{n+1}/p_n)p_{n+1}+kp_n}{(p_{n+1}/p_n)q_{n+1}+kq_n}=\frac{p_{n+1}^2+kp_n^2}{p_{n+1}q_{n+1}+kp_nq_n}=N/D,$$

where $N=(p'_{n+1})^2+k(p'_n)^2$ and $D=p'_{n+1}q'_{n+1}+kp'_nq'_n$ [dividing throughout by the common factor $(-k)^n$]. Therefore, $p/h=N/D$. Now set $a=p'_{n+1}$, $b=p'_n$, $c=q'_{n+1}$, and $d=q'_n$ in (12) and let $Q=(q'_{n+1})^2+k(q'_n)^2$. That identity then becomes

$$D^2+k=NQ. \tag{13}$$

Let $g = \text{g.c.d.}(N, D)$. We see from (13) that $g|k$. Since $N = pg$ and $\text{g.c.d.}(p, k) = 1$(by hypothesis), it follows that $g = 1$, so $N = p$ and $D = h$. However, we know that $[a_{n+1}, \ldots, a_1]_k = p_{n+1} / p_n = p'_{n+1} / p'_n$. Setting $x = p'_{n+1}$ and $y = p'_n$ completes the proof of (4) and (5).

Summary: Given the minimal positive solution of the congruence in (2), we have indicated an algorithm for generating solutions of (5). This construction involves a special type of c.f. (base *k*), as defined by (1). The conditions in (3) might, at first glance, seem unduly restrictive. It may be shown, however, that p / h may always be put into the desired c.f. form in (3), provided that integers *x* and *y* exist that satisfy (5). The proof of this assertion is left to the interested reader.

Setting $k = 1$ in the problem yields Serret's construction (1848), one of several known in the literature for finding the *unique* *x* and *y* such that $p = x^2 + y^2$, provided *p* is a prime with $p \equiv 1$ (mod 4). Also, for $k = 1$, the identity in (12) reduces to an identity attributable to Leonardo of Pisa (a.k.a. Fibonacci), such identity appearing in his *Liber Abaci* (1202).

Two examples illustrate the construction's applicability.

Example 1: Let $k = 3$ and $p = 757$. Note that

$$\left(\frac{-3}{757}\right) = \left(\frac{-3 + 4 \cdot 757}{757}\right) = \left(\frac{3025}{757}\right) = \left(\frac{55^2}{757}\right) = 1.$$

Hence, the minimal positive solution of the congruence $h^2 \equiv -3 \pmod{757}$ is $h = 55$. Without disclosing the logic of the following expansion, we may at least verify its accuracy:

$$\begin{aligned}
&757 / 55 = 13 + 42 / 55 = 13 + 3 / \theta_1; \\
&\theta_1 = 55 / 14 = 3 + 13 / 14 = 3 + 3 / \theta_2; \\
&\theta_2 = 42 / 13 = 0 + 3 / \theta_3; \\
&\theta_3 = 13 / 14 = 0 + 3 / \theta_4; \\
&\theta_4 = 42 / 13 = 3 + 3 / 13 = 3 + 3 / \theta_5; \\
&\theta_5 = 13.
\end{aligned}$$

Thus, $757 / 55 = [13, 3, 0, 0, 3, 13]_3$, which is of the desired form, with $n = 2$. Then the solutions of $x^2 + 3y^2 = 757$ are found by $x / y = [0, 3, 13]_3$. We find the successive convergents of this c.f.: $0/1$, $3/3$, and $39/42$. Hence, $x / y = 39 / 42 = 13 / 14$, so $x = 13$ and $y = 14$. As we may verify, $13^2 + 3 \cdot 14^2 = 757$.

Example 2: Let $k = -2$ and $p = 193$. Since

$$\left(\frac{2}{193}\right) = \left(\frac{2 + 193 \cdot 14}{193}\right) = \left(\frac{2704}{193}\right) = \left(\frac{52^2}{193}\right) = 1$$

we see that $h = 52$ is the minimal positive solution of the congruence $h^2 \equiv 2 \pmod{193}$. We may expand $193 / 52$ as follows:

$$\begin{aligned}
&193 / 52 = 5 - 67 / 52 = 5 - 2 / \theta_1; \\
&\theta_1 = 104 / 67 = 2 - 30 / 67 = 2 - 2 / \theta_2; \\
&\theta_2 = 67 / 15 = 5 - 8 / 15 = 5 - 2 / \theta_3; \\
&\theta_3 = 15 / 4 = 5 - 5 / 4 = 5 - 2 / \theta_4; \\
&\theta_4 = 8 / 5 = 2 - 2 / 5 = 2 - 2 / \theta_5; \\
&\theta_5 = 5.
\end{aligned}$$

Thus, $193/52 = [5, 2, 5, 5, 2, 5]_{-2}$, which is of the desired form, with $n = 2$. Therefore, solutions of $x^2 - 2y^2 = 193$ are found from $x/y = [5, 2, 5]_{-2}$. This yields the convergents: $5/1$, $8/2$, and $30/8$, so $x = 15$ and $y = 4$. Q.E.D.

Searching for Pairs

H-511 *Proposed by M. N. Deshpande, Aurangabad, India (Vol. 34, no. 2, May 1996)*

Find all possible pairs of positive integers m and n such that $m(m+1) = n(m+n)$. [Two such pairs are: $m = 1, n = 1$; $m = 9, n = 6$.]

Solution by H.-J. Seiffert, Berlin, Germany

The pairs $(m, n) \in N^2$ asked for are $(m, n) = (F_{2k}^2, F_{2k-1}F_{2k})$, where k is a positive integer. It is easily verified that, for these pairs, the considered equation is indeed satisfied.

Below we will use the well-known result that all solutions $(a, b) \in N^2$ of the Pell equation $a^2 - 5b^2 = -4$ are given by $(a, b) = (L_{2k-1}, F_{2k-1})$, $k \in N$. In particular, we have $a \geq b$.

Let $(m, n) \in N^2$ such that $m(m+1) = n(m+n)$. Write $m = rp$ and $n = rq$, where $p, q, r \in N$ such that $\gcd(p, q) = 1$. Then the given equation becomes $p(rp+1) = rq(p+q)$, which shows that r divides p. Letting $p = rs$, $s \in N$, we get $s(r^2s+1) = q(rs+q)$. From $p = rs$, $\gcd(p, q) = 1$, and $s|q^2$, it follows that $s = 1$. Now, the resulting equation $r^2 + 1 = q(r+q)$ may be written as $(2r-q)^2 - 5q^2 = -4$. Hence, $(2r-q, q) = (L_{2k-1}, F_{2k-1})$ for some $k \in N$. It readily follows that $r = F_{2k}$, so that we have $(m, n) = (F_{2k}^2, F_{2k-1}F_{2k})$.

Also solved by P. Bruckman, L. A. G. Dresel, A. Dujella, C. Georghiou, and the proposer.

FPP's

H-512 *Proposed by Paul S. Bruckman, Highwood, IL (Vol. 34, no. 2, May 1996)*

The *Fibonacci pseudoprimes* (or FPP's) are those composite n with g.c.d.$(n, 10) = 1$ such that $n|F_{n-\varepsilon_n}$, where ε_n is the Jacobi symbol $(\frac{5}{n})$. Suppose $n = p(p+2)$, where p and $p+2$ are "twin primes." Prove that n is a FPP if and only if $p \equiv 7 \pmod{10}$.

Solution by Lawrence Somer, Catholic University of America, Washington DC

We first suppose that $p \equiv 7 \pmod{10}$. Then $p+2 \equiv 9 \pmod{10}$. By quadratic reciprocity, we see that $(\frac{5}{p}) = -1$ and $(\frac{5}{p+2}) = 1$. Hence, $(\frac{5}{p(p+2)}) = (\frac{5}{p})(\frac{5}{p+2}) = (-1)(1) = -1$. We want to show that $p(p+2)|F_{p(p+2)+1}$. It is well known that $F_n|F_{kn}$ for any positive integer k. Since both p and $p+2$ are primes, $p|F_{p-\varepsilon_p} = F_{p+1}$ and $p+2|F_{p+2-\varepsilon_{p+2}} = F_{p+1}$. Further, since $p(p+2)+1 = (p+1)^2$, $F_{p+1}|F_{(p+1)^2}$, and g.c.d.$(p, p+2) = 1$, we see that $p(p+2)|F_{p(p+2)+1}$.

Now suppose that $n = p(p+2)$ is a FPP. We must have $p \equiv 1, 3, 5, 7$, or $9 \pmod{10}$. If $p \equiv 5 \pmod{10}$, then g.c.d.$(n, 10) \neq 1$. If $p \equiv 3 \pmod{10}$, then $p+2 \equiv 5 \pmod{10}$ and, again, g.c.d.$(n, 10) \neq 1$. Suppose $p \equiv 1 \pmod{10}$. Then $p+2 \equiv 3 \pmod{10}$. By quadratic reciprocity, $(\frac{5}{p}) = 1$ and $(\frac{5}{p+2}) = -1$. Hence, $\varepsilon_n = (\frac{5}{p(p+2)}) = (\frac{5}{p})(\frac{5}{p+2}) = (1)(-1) = -1$, so $n - \varepsilon_n = p(p+2)+1 = p^2+2p+1$. Thus, $p(p+2)|F_{p^2+2p+1}$. It is well known that $(F_a, F_b) = F_{(a,b)}$, where (a, b) denotes the g.c.d. of a and b. We note that $p|F_{p-\varepsilon_p} = F_{p-1}$. Now, $p^2+2p-3 = (p-1)(p+3)$. Hence,

$p|F_{p^2+2p-3}$. Therefore, $p|(F_{p^2+2p+1}, F_{p^2+2p-3})$, which implies that $p|F_{(p^2+2p+1,\, p^2+2p-3)}$. However, $(p^2+2p+1, p^2+2p-3)|(p^2+2p+1)-(p^2+2p-3)=4$, so $p|F_4=3$. Thus, $p=3$, which is a contradiction since $p\equiv 1 \pmod{10}$. Thus, $p\not\equiv 1 \pmod{10}$. Now suppose that $p\equiv 9 \pmod{10}$. Then $p+2\equiv 1 \pmod{10}$. By quadratic reciprocity, $(\frac{5}{p})=(\frac{5}{p+2})=1$. Therefore, $\varepsilon_n=(\frac{5}{p(p+2)})=(\frac{5}{p})(\frac{5}{p+2})=(1)(1)=1$, so $n-\varepsilon_n=p(p+2)-1=p^2+2p-1$. Now, $p|F_{p-\varepsilon_p}=F_{p-1}$. Thus, as in our above argument, $p|F_{p^2+2p-3}$. Hence, $p|(F_{p^2+2p-1}, F_{p^2+2p-3})=F_{(p^2+2p-1,\, p^2+2p-3)}$. However, $(p^2+2p-1, p^2+2p-3)|(p^2+2p-1)-(p^2+2p-3)=2$. Thus, $p|F_2=1$, which is a contradiction. Therefore, $p\not\equiv 9 \pmod{10}$; hence, $p\equiv 7 \pmod{10}$.

Also solved by L. A. G. Dresel, A. Dujella, H.-J. Seiffert, D. Terr, and the proposer.

Sum Product

H-513 ***Proposed by Paul S. Bruckman, Highwood, IL***
(Vol. 34, no. 4, August 1996)

Define the following quantities:

$$A=\sum_{n\ge 0}\frac{1}{(n!)^2},\quad B=\sum_{n\ge 0}\frac{1}{n!(n+1)!},\quad C=\sum_{n\ge 0}\frac{(2n)!}{(n!)^4},\quad D=\sum_{n\ge 0}\frac{(2n+2)!}{n!((n+1)!)^2(n+2)!}.$$

Prove that $A^2D=B^2C$.

Solution by the proposer

Clearly, the series defining A and B are convergent. Using Stirling's formula, $\binom{2n}{n}\sim 4^n(n\pi)^{-1/2}$ as $n\to\infty$. Thus, the convergence of the series defining C is comparable to that of the series

$$\sum_{n\ge 1}\frac{4^n}{n^{1/2}(n!)^2};$$

since the latter series is clearly convergent, so is the series defining C. Also, D is defined by a series that is comparable to the series

$$\sum_{n\ge 1}\frac{4}{n^2}\cdot\frac{(2n)!}{(n!)^4},$$

and so the series defining D is convergent. Clearly, all quantities are positive quantities.

We recognize the Modified Bessel Functions of integer order, defined as follows:

$$I_n(z)=\left(\tfrac{1}{2}z\right)^n\sum_{k\ge 0}\frac{\left(\frac{1}{4}z^2\right)^k}{k!(n+k)!},\ \text{an entire function of } z,\ n=0,1,2,\dots. \quad (1)$$

See, e.g., *Handbook of Mathematical Functions*, ed. M. Abramowitz & I. A. Stegun (9th prtg., §9. Washington, D.C.: National Bureau of Standards, 1970). We then see that $A=I_0\equiv I_0(2)$ and $B=I_1\equiv I_1(2)$. It is also indicated in this source that the following relation holds:

$$I_m(z)I_n(z)=\left(\tfrac{1}{2}z\right)^{m+n}\sum_{k\ge 0}\frac{(2k+m+n)!\left(\frac{1}{4}z^2\right)^k}{(k+m)!(k+n)!k!},\quad m,n=0,1,2,\dots. \quad (2)$$

It follows from (2) that $C=(I_0)^2$ and $D=(I_1)^2$. Then $A^2D=B^2C=(I_0I_1)^2$.

Also Solved by C. Georghiou.

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