FRACTAL CONSTRUCTION BY ORTHOGONAL PROJECTION USING THE FIBONACCI SEQUENCE

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1. INTRODUCTION

Let $\{G(j,k)\}_{k=1}^{\infty}$ denote the Fibonacci j-sequences such that $G(2,k)=F_k$, the k^{th} Fibonacci number and, for $j>2$,

Definition 1: $\{G(j,k)\}_{k=1}^{\infty}=\{G(j,k), k=1,2,\ldots. G(j,k)=G(j-1,k), k=1,2,\ldots,j.\ G(j,k)=G(j,k-1)+G(j,k-2)+\cdots+G(j,k-j), k>j\}$.

Thus, new elements of the set $\{G(j,k)\}_{k=1}^{\infty}$ for $j=2,3,\ldots$ are created by adding the previous j elements of the sequence, using as initial values the first j values of $\{G(j-1,k)\}_{k=1}^{\infty}$. Fibonacci j-sequences, satisfying the j^{th}-order linear recurrence relation in Definition 1, are also called j-bonacci, j-acci, or j-generalized Fibonacci numbers. They are a special case of a general linear recurrence relation studied by Levesque [10] and Tee [17]. The case $j=3$ yields so-called Tribonacci numbers (see Feinberg [6]). Table 1 gives the values $\{G(j,k)\}_{k=1}^{16}$ for $j=2,\ldots,7$.

TABLE 1. $\{G(j,k)\}_{k=1}^{16}$ for $j=2,\ldots,7$

j/k																
1/0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	1	1	2	3	5	8	13	21	34	55	89	144	233	377	610	987
3	1	1	2	4	7	13	24	44	81	149	274	504	927	1705	3136	5768
4	1	1	2	4	8	15	29	56	108	208	401	773	1490	2872	5536	10671
5	1	1	2	4	8	16	31	61	120	236	464	912	1793	3525	6930	13624
6	1	1	2	4	8	16	32	63	125	248	492	976	1936	3840	7617	15019
7	1	1	2	4	8	16	32	64	127	253	504	1004	2000	3984	7936	15808

There is associated with each sequence $\{G(j,k)\}_{k=1}^{\infty}$ the j^{th}-degree polynomial

$$F_j(x)=x^j-x^{j-1}-x^{j-2}-\cdots-x-1, \tag{1}$$

denoted in the present paper as Fibonacci j-polynomials. Let the real or complex number S_j denote the sum of the j^{th} powers of the roots of a polynomial of degree j. Then Newton's formula is given by (see Tee [18])

$$S_j=a_1S_{j-1}+a_2S_{j-2}+\cdots+a_{j-1}S_1+ja_j,\ S_1=1, \tag{2}$$

where the a_1 are coefficients of the monic polynomial $x^j-a_1x^{j-1}-a_2x^{j-2}-\cdots-a_{j-1}x-a_j=0$.

As an observation, referring to (1), $G(j, j+2) = 2^j - 1 = S_j$, $j \geq 2$ if $a_i = 1$, $\forall i$, which can be shown inductively. Godsil and Razen [8] derived the generating function for a self-generating sequence having parameters k, m, and r, denoted $\mathrm{SGS}(k, m, r)$, given by

$$F(x) = \frac{p_{k+r}(x)}{(1-x)^k - mx^{k+r}},$$

and showed no Fibonacci j-sequence was a SGS for $j \geq 4$, where p_{k+r} is a polynomial of degree at most $k+r$. The well-known generating function for Fibonacci j-sequences (see Philippou [15]),

$$Q_j(x) = \frac{x}{1 - x - x^2 - \cdots - x^{j-1} - x^j} = \sum_{k=1}^{\infty} G(j, k)x^k, \ |x| < 0.5, \tag{4}$$

which also appeared in the work of Godsil and Razen [8], is a special case of the generating formula of Levesque [10]. If x is replaced by η^{-1} and a factor of $\eta - 1$ is introduced, then (4) becomes

$$\sum_{k=1}^{\infty} \frac{G(j, k)}{\eta^k} = \frac{(\eta-1)\eta^{j-1}}{1+(\eta-2)\eta^j}, \ |\eta| > 2. \tag{5}$$

The region of convergence of (5) is (see Tee [17]) $\{\eta : |\eta| > x_j\}$, where x_j is the largest real root of (1). The form of (5) is useful in the context of the present paper. A derivation of (5) is presented in the next section by an alternate method that also reveals several number theoretic properties of the sequences $\{G(j, k)\}_{k=1}^{\infty}$. Properties of the zeros of the Fibonacci j-polynomials $F_j(x)$ are restated, and several are proved by a different method.

Another result of the paper is a geometrical interpretation of $\{G(j, k)\}_{k=1}^{\infty}$ in terms of a sequence of sets such that the first set depends on the Fibonacci numbers and subsequent sets on the Fibonacci j-sequences. For $j = 2$, a fractal is given and it is shown that a sequence of compact sets exists such that the fractal dimension, counting, and tiling features depend on the Fibonacci j-sequences. An exact expression for the fractal dimension is derived which depends on the largest real zeros of the Fibonacci j-polynomials, x_j, $\forall j \geq 2$. Fractals are of interest in the mathematical sciences (see Mandelbrot [12]).

2. CONVERGENCE PROPERTIES

Miller [13] showed that the zeros of the polynomials $F_j(x)$ are distinct, all but one lies in the unit disk and the latter is real and lies in the interval $(1, 2)$. Flores [7] showed that $x_j \to 2$ as $j \to +\infty$. The monotonic properties of the sequence $\{x_j\}_{j=1}^{+\infty}$ are indicated in

Lemma 1:

$$1 < x_j < x_{j+1} < 2, \ j = 2, 3, \ldots, \tag{6}$$

$$x_j \to 2 \text{ monotonically as } j \to +\infty. \tag{7}$$

Proof: Referring to (1), for each j, $F_j(1) = -(j-1)$, $F_j(2) = 1$. Thus, there is a real zero, denoted x_j. Since $F_j(x) - F_{j-1}(x) = x^{j-1}(x-2)$, it follows by continuity that

$$F_j(x) < F_{j-1}(x), \ 0 < x < 2,$$
$$F_j(x) > F_{j-1}(x), \ 2 < x < +\infty,$$

which implies (6). Note that x_j is largest in magnitude among zeros, since $F_j(x) > F_2(x) > 1$ if $x > 2, j > 2$. To show (7), write

$$F_j(x) = F_2(x) + x^2(x-2)\frac{1-x^{j-2}}{1-x}.$$

Suppose that $\sup\{x_j : F_j(x_j) = 0\} = \varepsilon < 2$. If $\delta_j \to 0^+$ as $j \to +\infty$ then, for some positive sequence $\{\delta_j\}_{j=1}^{+\infty}$, it follows by continuity of F_j that the δ_j may be chosen small enough that $|F_j(\varepsilon - \delta_j)| < 1/j$. Thus, noting that $\varepsilon > 1$, it follows that

$$\lim_{j\to+\infty} F_j(\varepsilon-\delta_j) = \lim_{j\to+\infty}\left(F_2(\varepsilon-\delta_j) + (\varepsilon-\delta_j)^2(\varepsilon-\delta_j-2)\frac{1-(\varepsilon-\delta_j)^{j-2}}{1-\varepsilon+\delta_j}\right) = -\infty.$$

The previous statement is a contradiction which proves the lemma.

A result of Flores [7] is the following theorem.

Theorem 1: For sufficiently large *k*, $\exists$ a constant $c > 0$ such that

$$G(j,k) \approx cx_j^k \quad \text{and} \quad \lim_{k\to+\infty}\frac{G(j,k+1)}{G(j,k)} = x_j,\ j = 2, 3, \ldots. \tag{8}$$

The following numerical examples were calculated on a *TI*–85©:

$$x_3 = 1.839286755,\ x_4 = 1.927561975,\ x_{10} = 1.999018633,\ x_{20} = 1.999999046.$$

The exponential growth of the Fibonacci *j*-sequences is evident in

Corollary 1: Let $M > 0, n \in Z^+$. Then $\forall_j > 2,\ 1 \le i < j-1,\ \exists k$ such that

$$|G(j,k_0) - MG(j-i,k_0)| > n,\ k_0 > k. \tag{9}$$

Proof: By Theorem 1 and the definition of limit, there is a constant $C > 0$ that depends on i, j so that, for large enough *k*, $G(j,k)/G(j-i,k) > C(x_j/x_{j-i})^k \to +\infty$ as $k \to +\infty$.

In [17] Tee showed convergence of the infinite series in the following theorem as a special case of a more general result if $|\eta| > x_j,\ j = 2, 3, \ldots$, for which a proof is also given in the present paper.

Theorem 2:

$$G_j(\eta) = \sum_{k=1}^{\infty}\frac{G(j,k)}{\eta^k} = \frac{(\eta-1)\eta^{j-1}}{1+(\eta-2)\eta^j},\ |\eta| > x_j,\ j = 2, 3, \ldots, \tag{10}$$

such that (10) diverges at $\eta = \pm x_j$ and

$$\lim_{j\to+\infty} G_j(\eta) = \begin{cases}(\eta-1)/((\eta-2)\eta), & \text{if } \eta > 2,\ \eta \le -2,\\ +\infty, & \text{if } \eta = 2.\end{cases} \tag{11}$$

Proof: The theorem is proved first for $\eta = 2,\ j \ge 2$. A sketch of the proof, which is essentially the same as that for $\eta = 2$, is indicated for values of η other than 2. Parallel results are given for $\eta < -x_j$, for which the same method is applicable. A sequence of lemmas establishes the theorem for values of $j \geq 2$. Define the infinite sequence

$$H(1, j, k) = G(j, k+3) - \left(\sum_{i=1}^{k+1} G(j, i) + 1 \right), \; k \geq 1. \tag{12}$$

The significance of the 1 in the argument of H will become apparent. Then

Lemma 2:

$$\frac{1}{2} \sum_{k=1}^{\infty} \frac{G(j, k)}{2^k} = 1 + \sum_{k=1}^{\infty} \frac{H(1, j, k)}{2^{k+3}}. \tag{13}$$

$$\frac{5}{6} \sum_{k=1}^{\infty} \frac{G(j, k)(-1)^k}{2^k} = -\frac{1}{3} + \sum_{k=1}^{\infty} \frac{H(1, j, k)(-1)^{k+1}}{2^{k+3}}. \tag{14}$$

Proof: Equation (14) corresponds to $\eta = -2$. To prove (13), expand

$$\sum_{k=1}^{\infty} \frac{G(j, k)}{2^k} = \frac{1}{2} + \frac{1}{4} + \cdots + \frac{G(j, k)}{2^k} + \cdots$$

$$= \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^k} + \cdots + \frac{G(j, 3) - 1}{8} + \frac{G(j, 4) - 1}{16} + \cdots + \frac{G(j, k) - 1}{2^k} + \cdots$$

$$= \sum_{k=1}^{\infty} \frac{1}{2^k} + \left(\frac{1}{8} + \frac{1}{16} + \cdots \right) G(j, 1) + \frac{G(j, 3) - 1 - G(j, 1)}{8} + \frac{G(j, 4) - 1 - G(j, 1)}{16} + \cdots$$

$$+ \frac{G(j, k) - 1 - G(j, 1)}{2^k} + \cdots$$

$$= 1 + \frac{1}{2} \frac{G(j, 1)}{2} + \left(\frac{1}{16} + \frac{1}{32} + \cdots \right) G(j, 2) + \frac{G(j, 4) - 1 - G(j, 1) - G(j, 2)}{16} + \cdots$$

$$+ \frac{G(j, k) - 1 - G(j, 1) - G(j, 2)}{2^k} + \cdots$$

$$= \cdots = 1 + \frac{1}{2} \frac{G(j, 1)}{2} + \frac{1}{2} \frac{G(j, 2)}{4} + \frac{1}{2} \frac{G(j, 3)}{8} + \frac{H(1, j, 1)}{16} + \frac{H(1, j, 2)}{32} + \cdots$$

$$+ \frac{G(j, k) - 1 - G(j, 1) - G(j, 2) - G(j, 3)}{2^k} + \cdots \forall k > 5$$

$$= \cdots = 1 + \frac{1}{2} \frac{G(j, 1)}{2} + \frac{1}{2} \frac{G(j, 2)}{4} + \frac{1}{2} \frac{G(j, 3)}{8} + \cdots + \frac{1}{2} \frac{G(j, k+1)}{2^{k+1}}$$

$$+ \frac{H(1, j, 1)}{16} + \frac{H(1, j, 2)}{32} + \cdots + \frac{H(1, j, k)}{2^{k+3}}$$

$$+ \frac{G(j, k') - 1 - G(j, 1) - G(j, 2) - \cdots - G(j, k+1)}{2^{k'}} + \cdots,$$

for every $k' > k+3$. Denote the last term in the above expression by $I(k, j, k') / 2^{k'}$, $\forall k' > k+3$, $k \geq 1$.

By the definition of $G(j,k)$ and $H(1,j,k)$

$$G(j,k')/2^{k'} > I(k,j,k')/2^{k'} > H(1,j,k)/2^{k'} \geq 0,\ j \geq 2. \tag{15}$$

Thus, by Theorem 1 and (15), the last term approaches 0 as k and $k' \to +\infty$ and (13) follows. The proof of (14) is similar, and one obtains

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{G(j,k)(-1)^k}{2^k} &= \frac{-1}{2}+\frac{1}{4}+\cdots+\frac{(-1)^k G(j,k)}{2^k}+\cdots\\ &= \frac{-1}{3}-\frac{1}{6}\frac{G(j,1)}{2}+\frac{1}{6}\frac{G(j,2)}{4}-\frac{1}{6}\frac{G(j,3)}{8}+\cdots+\frac{(-1)^{k+1}}{6}\frac{G(j,k+1)}{2^{k+1}}\\ &\quad+\frac{H(1,j,1)}{16}-\frac{H(1,j,2)}{32}+\cdots+\frac{(-1)^{k+1}H(1,j,k)}{2^{k+3}}\\ &\quad+(-1)^{k'}\frac{G(j,k')-1-G(j,1)-G(j,2)-\cdots-G(j,k+1)}{2^{k'}}+\cdots.\end{aligned}$$

Noting that $H(1,2,k)=0$, $\forall k \geq 1$, which follows from (12), and the identity $F_1+F_2+\cdots+F_n=F_{n+2}-1$, define the following infinite sequences depending on j and k:

$$H(i,j\,k) = H(i-1,j,k+1)-G(j,k+2),\ i=2,\ldots,j-2,\ j\geq 4, \tag{16}$$

$$H(j-1,j,k) = H(j-2,j,k+3)-G(j,k+4),\ k\geq 1,\ j\geq 3. \tag{17}$$

Note that (16) begins at $j=4$ and (17) begins at $j=3$. Then, for $j\geq 3$, we have

Lemma 3:

$$\begin{aligned}H(j-1,j,k) &= G(j,k+j+3)-\left(\sum_{i=1}^{k+j+1}G(j,i)+1\right)\\ &\quad - G(j,k+j+1)-G(j,k+j)-\cdots-G(j,4) = H(1,j,k).\end{aligned} \tag{18}$$

Proof: By (16) and (17) [one can also use the identity $G(j,k+j+3)=2G(j,k+j+2)-G(j,k+2)$],

$$\begin{aligned}H(j-1,j,k) &= H(j-2,j,k+3)-G(j,k+4)\\ &= H(j-3,j,k+4)-G(j,k+5)-G(j,k+4)\\ &= \cdots = H(j-i,j,k+i+1)-G(j,j,k+i+2)-\cdots-G(j,k+4)\\ &= \cdots = H(1,j,k+j)-G(j,k+j+1)-\cdots-G(j,k+4),\ (i=j-1),\\ &= G(j,k+j+3)-\left(\sum_{i=1}^{k+j+1}G(j,i)+1\right)-G(j,k+j+1)-\cdots-G(j,k+4)\\ &= H(1,j,k)+G(j,k+j+3)-2(G(j,k+j+1)+G(j,k+j)+\cdots\\ &\quad +G(j,k+4))-G(j,k+3)-G(j,k+2)\\ &= H(1,j,k)+G(j,k+j+2)-(G(j,k+j+1)+G(j,k+j)+\cdots\\ &\quad +G(j,k+3)+G(j,k+2))\\ &= H(1,j,k).\end{aligned}$$

From (16)-(18) and (12),

$$H(i,j,k)=\begin{cases}H(i+1,j,k-1)+G(j,k+1), & \text{if } k>1, i=1,\ldots,j-3,\\ G(j,k+1)=1, & \text{if } k=1,\end{cases} \tag{19}$$

$$H(j-2,j,k)=\begin{cases}H(1,j,k-3)+G(j,k+1), & \text{if } k\ge 4,\\ G(j,k+1)=1,2,4, & \text{if } k=1,2,3, \text{ resp.}\end{cases} \tag{20}$$

The second result of (19) is shown as follows. By (12), $H(1,j,1)=1$. For $2\le m\le j-2$, $j\ge 4$,

$$\begin{aligned}H(m,j,1)&=H(m-1,j,2)-G(j,3)\\&=H(m-2,j,3)-G(j,4)-G(j,3)\\&=\cdots=H(1,j,m)-G(j,m+1)-\cdots-G(j,3)\\&=G(j,m+3)-\left(\sum_{i=1}^{m+1}G(j,i)+1\right)-G(j,m+1)-\cdots-G(j,3)\\&=G(j,m+3)-(1+G(j,1)+G(j,2)+2G(j,3)+\cdots+2G(j,m+1))\\&=2^{j-i-1}-2(1+2+\cdots+2^{j-i-3})-1,\ 0\le i\le j-4,\\&=2^{j-i-1}-(1+2+\cdots+2^{j-i-2})=1.\end{aligned}$$

The second result of (20) follows by a similar method. From (19) and (20), one obtains

Lemma 4:

$$\sum_{k=1}^{\infty}\frac{H(i,j,k)}{2^{k+2+i}}=\sum_{k=2}^{\infty}\frac{G(j,k)}{2^{k+1+i}}+\sum_{k=1}^{\infty}\frac{H(i+1,j,k)}{2^{k+3+i}},\ i=1,\ldots,j-3, \tag{21}$$

$$\sum_{k=1}^{\infty}\frac{H(j-2,j,k)}{2^{k+j}}=\sum_{k=2}^{\infty}\frac{G(j,k)}{2^{k+j-1}}+\sum_{k=1}^{\infty}\frac{H(1,j,k)}{2^{k+3+j}}. \tag{22}$$

Proof: The equalities (21) and (22) follow by summing (19) and (20) and adjusting the summation subscripts after division, respectively, by 2^{k+2+i} and 2^{k+j}.

Returning now to the proof of Theorem 2 when $\eta=2$, applying (21) and (22) in Lemma 4 recursively, it follows that

$$\frac{2^j-1}{2^j}\sum_{k=1}^{\infty}\frac{H(1,j,k)}{2^{k+3}}=\sum_{k=2}^{\infty}\frac{G(j,k)}{2^{k+2}}\left(1+\frac{1}{2}+\cdots+\frac{1}{2^{j-3}}\right). \tag{23}$$

Taking this and Lemma 2, one obtains

Lemma 5:

$$\sum_{k=1}^{\infty}\frac{G(j,k)}{2^k}=2^{j-1},\quad \sum_{k=1}^{\infty}\frac{G(j,k)(-1)^k}{2^k}=\frac{2^{j-1}3(-1)^j}{1-4(-2)^j}. \tag{24}$$

Proof: Substituting (13) into (23) yields

$$\frac{2^j-1}{2^j}\left(\frac{1}{2}\sum_{k=1}^{\infty}\frac{G(1,k)}{2^k}-1\right)=\frac{1}{4}\sum_{k=2}^{\infty}\frac{G(j,k)}{2^k}\left(1+\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^{j-3}}\right)$$

$$\Leftrightarrow\sum_{k=1}^{\infty}\frac{G(j,k)}{2^k}\left[\frac{2^j-1}{2^{j+1}}-\frac{1}{2}\left(1-\frac{1}{2^{j-2}}\right)\right]=\frac{2^j-1}{2^j}-\frac{1}{4}\left(1-\frac{1}{2^{j-2}}\right)$$

$$\Leftrightarrow\sum_{k=1}^{\infty}\frac{G(j,k)}{2^k}\left[\frac{2^j-1+4-2^j}{2^{j+1}}\right]=\frac{2^j-1-2^{j-2}+1}{2^j}$$

$$\Leftrightarrow\sum_{k=1}^{\infty}\frac{G(j,k)}{2^k}\frac{3}{2^{j+1}}=\frac{2^{j-2}3}{2^j}\Leftrightarrow\sum_{k=1}^{\infty}\frac{G(j,k)}{2^k}=2^{j-1}.$$

For $\eta=-2$ (and $\eta<-x_j$), the analogs of (21) and (22) are multiplied by $(-1)^k$ and $(-1)^{k+1}$ on the left- and right-hand sides, respectively. Similarly, by Lemmas 2 and 4,

$$\frac{2^j-(1)^j}{2^j}\sum_{k=1}^{\infty}\frac{H(1,j,k)(-1)^k}{2^{k+3}}=\sum_{k=2}^{\infty}\frac{G(j,k)(-1)^k}{2^{k+2}}\left(-1+\frac{1}{2}+\cdots+\frac{(-1)^j}{2^{j-3}}\right)$$

$$\Rightarrow\sum_{k=1}^{\infty}\frac{G(j,k)(-1)^k}{2^k}=\frac{2^{j-1}3(-1)^j}{1-4(-2)^j}.$$

A similar analysis yields the theorem for other values of η and these details are briefly outlined. Lemma 2 becomes

Lemma 6:

$$\left(1-\frac{1}{\eta(\eta-1)}\right)\sum_{k=1}^{\infty}\frac{G(j,k)}{\eta^k}=\frac{1}{\eta-1}+\sum_{k=1}^{\infty}\frac{H(1,j,k)}{\eta^{k+3}},\ \eta>x_j, \tag{25}$$

$$\left(1-\frac{1}{\eta(\eta+1)}\right)\sum_{k=1}^{\infty}\frac{G(j,k)(-1)^k}{\eta^k}=\frac{-1}{\eta+1}+\sum_{k=1}^{\infty}\frac{H(1,j,k)(-1)^{k+1}}{\eta^{k+3}},\ \eta>x_j. \tag{26}$$

Proof: This result follows by straightforward application of geometric series. The interval of convergence follows by an argument similar to that given for Lemma 2.

Lemma 4 and (23) have 2 everywhere replaced by η. By Theorem 1 and the ratio test for absolute convergence, (10) diverges at the endpoints $\eta=\pm x_j$ and, therefore, diverges for $\{\eta:\eta\le|x_j|\}$. To prove (11), observe that $G_j(2)=2^{j-1}$ which, as $j\to+\infty$, implies the second part of (11). The first part follows directly by factoring η^{j-1} from $G_j(\eta)$ and letting $j\to+\infty$. The infinite series (10) is absolutely convergent for all values of j. Table 2 gives the values of $\{H(i,j,k)\}_{k=1}^{16}$, $i=1,2,3,4$, and $G(j,k)$ for $j=5$. The sequences $\{H(i,j,k)\}_{k=1}^{\infty}$ appear as periodic differences, as defined in (16) and (17).

In the next section, a brief introduction to fractals and fractal dimension is given along with several examples of fractals. A fractal is presented with counting features depending on the Fibonacci numbers.

TABLE 2. $\{G(5,k),\ H(i,j,k)\}_{k=1}^{16}$ **for** $i = 1, 2, 3, 4$

$i/j/k$																
$k =$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$G(5,k)$	1	1	2	4	8	16	31	61	120	236	464	912	1793	3525	6930	13624
$i = 1$	1	3	7	14	28	56	111	219	431	848	1668	3280	6449	12679	24927	49006
2	1	3	6	12	25	50	99	195	384	756	1487	2924	5749	11303	22222	43678
3	1	2	4	9	19	38	75	148	292	575	1131	2224	4373	8598	16894	33223
4	1	3	7	14	28	56	111	219	431	848	1668	3280	6449	12679	24927	49006

3. FIBONACCI NUMBERS AND FRACTALS

By definition, a fractal is a self-similar (self-affine) structure such that the topological dimension is strictly less than the Hausdorff-Besicovitch dimension (see Mandelbrot [12]). The topological "covering" dimension D_T or a set X has the property that any open covering of X has an open refinement with at most $D_T + 1$ open sets intersecting (see Hastings & Sugihara [9]). $D_T = 2$ in $\Re^2$, since any open covering has a refinement with at most three open sets intersecting.

Another important concept in fractals is the Box dimension

$$D = \lim_{r \to 0} \frac{-\log N(r)}{\log r} \tag{27}$$

where, for each $r > 0$, $N(r)$ is the smallest number of open balls having radius r which also cover X (see [9]). D is also denoted as the Hausdorff dimension when the dimensions are equal, including the fractal in the present paper. The value of D is computed for simple geometrical objects using the concept of scale factor and scaling dimension. Suppose X is reconstructed into n scaled copies of itself, each diminished in size by a factor k. Then

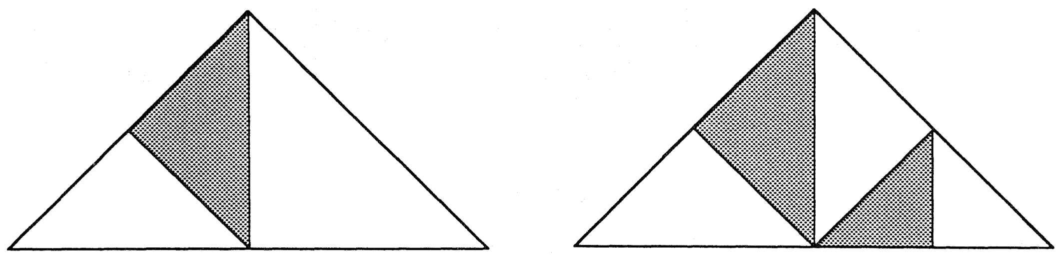
$$D = \frac{\log n}{\log 1/k}. \tag{28}$$

In certain fractals the scaling, Hausdorff, and Box dimension are all equal, including the fractals in the present work, since the Hausdorff dimension of a self-similar set with scaling ratio $1/k$ satisfies (28) also (see Crownover [5]).

Fractals are generated mathematically and have a geometric structure in Euclidean space. They are used as mathematical models for natural objects such as length of shorelines, leaf or fern patterns, Brownian motion, chaos, cause and effect such as minimization of energy to create fractal-like mud-flats, more exotically, minimization of scalar fields in the self-reproducing inflationary universe. The artist M. C. Escher was a precursor to many geometric ideas, having created drawings of self-similar structures (see *Scientific American* [16]). Fractals have also been studied by Pietgen, Jürgens, and Saupe (see [14]) who give an interesting introduction to the subject.

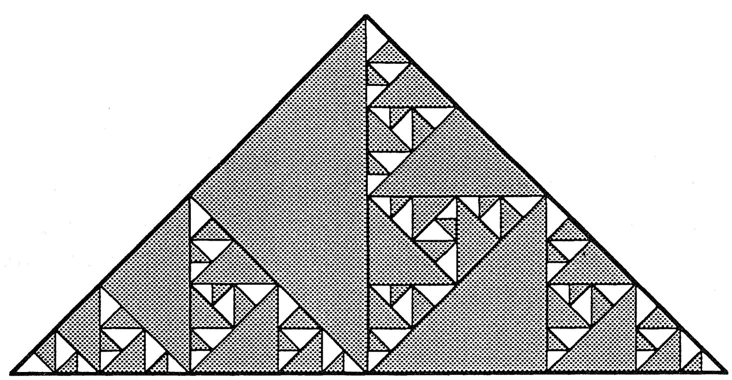
The concept of a self-similar (self-affine) structure is intrinsic to a fractal, although not all self-similar structures are fractals, i.e., continually subdividing a square into four sub-squares does not create a fractal because $D = D_T = \log 2^{2n} / \log 2^n = 2$.

The Cantor set is defined by removing the middle third of a given set of intervals, starting with the unit interval. The Cantor set has the cardinality of the unit interval although it is a totally disconnected set. In the present work a self-affine, two-dimensional structure is created by beginning with a right triangle and then orthogonally projecting onto the sides in clockwise direction [Figs. 1(a), 1(b), 2(a), and 2(b)].

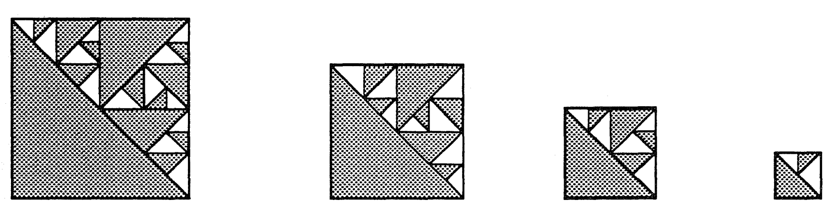


(a) Fractal Generation, Level 1, $j = 2$ **(b) Fractal Generation, Level 2, $j = 2$**

FIGURE 1



(a) Fractal Generation, Level 8, $j = 2$



(b) Self-Similar Pieces

FIGURE 2

In the present paper the geometrical meaning of "orthogonally project" (on the sides of an isosceles right triangle) is as follows: begin with the vertex of the right angle and draw a line that is orthogonal to and meets the opposite side at the midpoint. Proceeding clockwise, from the midpoint of this side draw another line orthogonally to the midpoint of the opposite side. The new boundaries form a right triangle that is similar to the original triangle but rotated 90 degrees counterclockwise. Likewise, two other similar triangles are formed, one on either side of the new triangle. It is left only to decide which triangles the process is applied to at every stage.

To give an example: begin with a right triangle with vertices at $(0, 1)$, $(-1, 0)$, $(1, 0)$. Let this be the 0^{th} stage. At stage $k = 1$, orthogonally project onto the sides of the triangle in a clockwise direction. This forms a triangle (grey-shaded) and two new (unshaded) triangles. Continue this process by orthogonally projecting onto the *largest* of the unshaded triangles at level k, $k = 1, 2, \dots$.

Lemma 7: This process forms a Fibonacci sequence such that the total number of unshaded triangles at level k is F_{k+2} of which F_{k+1} are largest. The total number of shaded triangles at stage k is $F_{k+2}-1$. The respective side lengths of the largest unshaded triangles is reduced in side length for consecutive stages by scale factor $1/\sqrt{2}$.

Proof: By inspection of Figure 1(a), the induction hypothesis is true for $k = 1$, since $F_3 = 2$. At level k, there are F_{k+2} unshaded triangles of which F_{k+1} are largest and scaled in length size by a factor of $1/\sqrt{2}$ with respect to stage $k-1$. Projecting on the F_{k+1} largest triangles results in $2F_{k+1} + F_k = F_{k+3}$ unshaded triangles of which $F_k + F_{k+1} = F_{k+2}$ are largest, since the scale factor of the largest to the smallest at any stage is $1/\sqrt{2}$. Thus, the small unshaded triangles at stage k become some of the large unshaded triangles at stage $k+1$. That the number of shaded triangles is $F_{k+2}-1$ follows by induction also, since at level $k+1$ the number of old and new shaded triangles is $(F_{k+2}-1) + F_{k+1} = F_{k+3}-1$.

The large unshaded triangles in Figures 1 and 2 and described in Lemma 7 are generated by an affine transformation of the form

$$\mathcal{T}(x, y) = \frac{1}{\sqrt{2}}\begin{bmatrix}\cos\frac{3\pi}{4} & -\sin\frac{3\pi}{4}\\ \sin\frac{3\pi}{4} & \cos\frac{3\pi}{4}\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} + \begin{bmatrix}a_{k,i}\\ b_{k,i}\end{bmatrix},$$

where k is the level of the projection and the integer i depends on k. Determining the precise values for $a_{k,i}$ and $b_{k,i}$ are not considered in the present paper. The small unshaded triangles in Figures 1 and 2 and described in Lemma 7 are generated by the following affine transformation:

$$\mathcal{U}(x, y) = \frac{1}{2}\begin{bmatrix}\cos 0 & -\sin 0\\ \sin 0 & \cos 0\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} + \begin{bmatrix}c_{k,i}\\ d_{k,i}\end{bmatrix},$$

for real numbers $c_{k,i}$, $d_{k,i}$, $k = 1, 2, 3, \dots$. For example, $a_{1,1} = b_{1,1} = 1/2$, $c_{1,1} = -1/2$, $d_{1,1} = 0$.

Construct a compact set E_2 as follows: at each stage k, for a projection on a given triangle, delete the shaded triangle, leaving two open unshaded triangles and their boundaries. Let $\mathcal{W}_k$ denote the union of all of the unshaded triangles with their boundaries at level k. Then set $E_2 = \bigcap_{k=1}^{+\infty}\mathcal{W}_k$. Clearly, E_2 is self-similar by construction, that is, E_2 is scale invariant and has topological dimension 1, i.e., no open subsets of $\mathcal{R}^2$. Let L_{k+1} denote the perimeters of the large unshaded triangles in Lemma 7 corresponding to level k, which have hypotenuse length $1/\sqrt{2}^{k-2}$ and base=height length $1/\sqrt{2}^{k-1}$. It follows that $L_{k+1} = 2/\sqrt{2}^{k-1} + 1/\sqrt{2}^{k-2}$. Hence, observing that $\mathcal{W}_k \subseteq \mathcal{W}_{k-1}$, $\forall k \geq 2$, the total length, denoted L, of the set E_2 is given by

Lemma 8: $L = \lim_{k\to+\infty}(F_{k+1}L_{k+1} + F_k L_k) = +\infty$.

Proof: The proof follows from Theorem 1 and the fact that $\sqrt{2} < x_j$, $\forall j \geq 2$.

It is noted that Figure 2(a) contains eight basic shapes, a right, isosceles triangle T_0, Figure 4 rotated counterclockwise through 135°. The shapes are generated with the affine transformations $\mathcal{T}$ and $\mathcal{U}$.

Theorem 3: The compact set E_2 is a self-similar set with Box=scaling=Hausdorff (dimension) $D = 1.38848$.

Proof: Such sets are normally called fractals in the literature. The Hausdorff and Box dimensions both equal the scaling dimension, since E_2 is self-similar with two scaling ratios, $1/\sqrt{2}$ and $1/2$, as observed by the geometry of Figure 2(a). This is also evident in the transformations $\mathcal{T}(x, y)$ and $\mathcal{U}(x, y)$. By Lemma 7, there are F_k large triangles at level $k-1$, and the k^{th} Fibonacci number is almost linearly proportional to x_2^k for large k so that the scaling dimension

$$D = \lim_{k\to+\infty}\frac{\log cx_2^k}{\log\sqrt{2}^k} = \lim_{k\to+\infty}\frac{\log c + k\log x_2}{k\log\sqrt{2}}$$

$$= \frac{2\log x_2}{\log 2} = \frac{2(\log(1+\sqrt{5}) - \log 2)}{\log 2} = 1.38848.$$

For completeness, D is calculated by the Box Counting Theorem (see Barnsley [3]).

Theorem 4 (The Box Counting Theorem): Cover $\mathcal{R}^2$ by boxes of size Cr^n, $C > 0$. $0 < r < 1$, where C and r are fixed real numbers. Let $\mathcal{N}_n$ denote the number of boxes of side length Cr^n that intersect any compact set $\mathcal{H} \subseteq \mathcal{R}^2$. Then $\mathcal{H}$ has fractal dimension

$$D = \lim_{n\to+\infty}\frac{\log \mathcal{N}_n}{\log Cr^n}.$$

By inspection of Figure 2(a) and generalizing to all values of k, one finds that

Lemma 9: For $k = 1, 2, 3, \ldots, F_{k+4}$ squares of side length $1/2\sqrt{2}^{k-1}$ cover E_2.

Proof: To prove this lemma, we observe several facts: the triangles are all oriented with respect to the x–y plane so that either the sides or diagonals of any triangle (shaded or unshaded) are parallel to the y-axis. Hence, any covering or tiling of E_2 or a subset of E_2. can be done in two ways, which is countable if the covering boxes are aligned with the boundaries of the triangles. A right isosceles triangle of hypotenuse length $1/\sqrt{2}^{k-2}$ can be covered by two squares of side length $1/2\sqrt{2}^{k-2}$ or three squares of side length $1/2\sqrt{2}^{k-1}$ (see Fig. 3). By inspection of Figure 1(b), the two large unshaded triangles of hypotenuse length 1 can be covered by three boxes of side length $1/2\sqrt{2}$. The small unshaded triangle of hypotenuse length $1/\sqrt{2}$ can be covered by two boxes of the same side length ($1/2\sqrt{2}$). The total number of boxes is 8. For this value of k, the boxes can be oriented 45°, with diagonals parallel to the y-axis. Proceeding inductively, assume that F_{k+1} large and F_k small unshaded triangles of hypotenuse lengths $1/\sqrt{2}^{k-2}$ and $1/\sqrt{2}^{k-1}$, respectively, are generated that can be covered by $3F_{k+1}+2F_k=F_{k+4}$ squares of side length $1/2\sqrt{2}^{k-1}$. In applying the inductive hypothesis to $k+1$, we apply the identity $3F_{k+2}+2F_{k+1}=F_{k+5}$, the new unshaded triangles have hypotenuse length $1/\sqrt{2}^{k-1}$, $1/\sqrt{2}^{k}$, and the fact that $E_2\subseteq\mathcal{W}_k$, $\forall k\geq 1$. There is no overlap of covering boxes on adjacent unshaded triangles, since opposite to the hypotenuse of any unshaded triangle is the boundary or a shaded triangle of equal or greater area. The fractal dimension is given by the Box Counting Theorem:

$$D=\lim_{k\to+\infty}\frac{\log F_{k+4}}{\log 2\sqrt{2}^{k-1}}=1.38848.$$

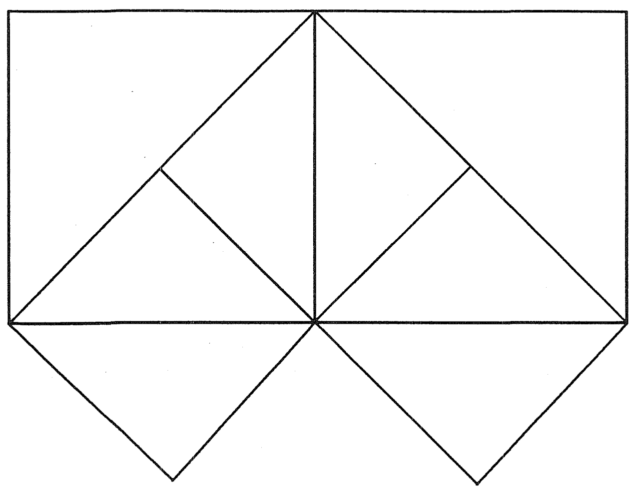


FIGURE 3. Covering the Right Isosceles Triangle by Squares

In the next section, triangles T_{j-2}, $j\geq 2$, are defined in the x–y plane. A theorem is given related to tiling $\bigcup_{i=0}^{j-2}T_i$, $j\geq 2$, with the triangles of the tiling enumerated by a Fibonacci j-sequence. E_j, $j\geq 2$, is characterized precisely, in terms of the union of a set of points that is contained in the set $T_0\cup T_1\cup\cdots\cup T_{j-2}=\bigcup_{i=0}^{j-2}T_i$. For a particular tiling of $\bigcup_{i=0}^{j-2}T_i$, it is shown that E_j is compact. In this case, the geometric object E_2 is translated, contracted in size, and rotated to create sets E_j, $j\geq 3$.

4. SETS E_j WITH FRACTAL DIMENSION

Consider the line $y=x+1$ and the ordinates $\{-1,1,3,7,\ldots,2^n-1,\ldots\}$ (Fig. 4). Denote by T_1, $T_3,\ldots,T_{2n-1}$ the triangles with boundaries formed by the set of vertices given, respectively, by

$$\{\{(1,0),(1,2),(0,1)\},\ \{(3,0),(3,4),(1,2)\},\ \{(7,0),(7,8),(3,4)\},\ldots,\\ \{(2^n-1,0),(2^n-1,2^n),(2^{n-1}-1,2^{n-1})\},\ldots\}.$$

Similarly, denote by $T_0, T_2, \ldots, T_{2n}$ the interlocking triangles with boundaries formed by the set of vertices given, respectively, by

$$\{\{(-1,0),(0,1),(1,0)\},\ \{(1,0),(1,2),(3,0)\},\ \{(3,0),(3,4),(7,0)\},\ldots,\\ \{(2^n-1,0),(2^n-1,2^n),(2^{n+1}-1,0)\},\ldots\}.$$

Note that T_0 does not follow the pattern given by the general triple of vertices.

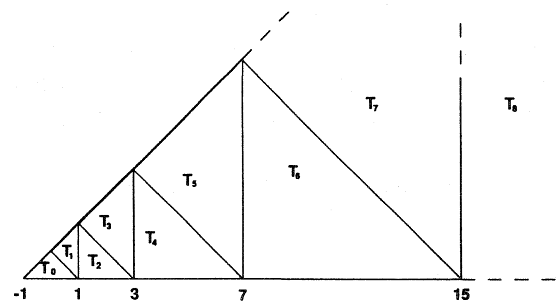


FIGURE 4. Fractal Boundaries

It follows that (if $\cup$ denotes the union of geometric objects) $T_0 \cup T_1$ is a reflection of T_0 about the line $y = 1 - x$, union with T_0; $T_0 \cup T_1 \cup T_2$ is a reflection of $T_0 \cup T_1$ about $x = 1$, union with $T_0 \cup T_1$; recursively, $T_0 \cup T_1 \cup \cdots \cup T_{2n-1}$ is a reflection of $T_0 \cup T_1 \cup \cdots \cup T_{2n-2}$ about the line $y = 2^n - 1 - x$, union with $T_0 \cup T_1 \cup \cdots \cup T_{2n-2}$; $T_0 \cup T_1 \cup \cdots \cup T_{2n}$ is a reflection of $T_0 \cup T_1 \cup \cdots \cup T_{2n-1}$ about the line $x = 2^n - 1$, union with $T_0 \cup T_1 \cup \cdots \cup T_{2n-1}$.

Theorem 5: A right isosceles triangle of area 2^{j-2} can be subdivided into (tiled by) similar triangles enumerated by the Fibonacci j-sequences $G(j,k)$, $\forall j \geq 2$. The total number of triangles of a given area forms a sequence $1, 1, 2, \ldots, G(j,k), \ldots$. The numbers in the sequence correspond with the number of similar triangles of area, respectively, 1/4, 1/8, ..., $G(j,k)/2^{k+1}, \ldots$.

Proof: Consider a right isosceles triangle of area 2^{j-2}, $j \geq 2$, for example $\bigcup_{i=0}^{j-2} T_i$, which can be subdivided into 2^j congruent subtriangles each having area 1/4. Figure 5 is a tiling of T_0 with 16 subtriangles each having area 1/16.

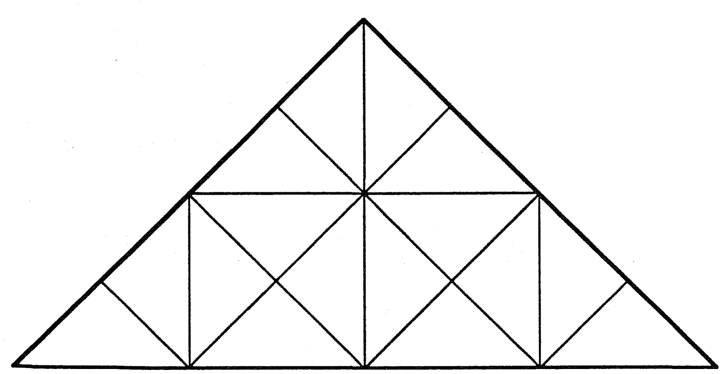


FIGURE 5. A Tiling of T_0

Construct another tiling of $\bigcup_{i=0}^{j-2} T_i$ as follows. The first element or triangle of the tiling consists of any one, $G(j,1)$, of the 2^j subtriangles. Then subdivide the remaining 2^j-1 triangles into $2^{j+1}-2$ similar subtriangles of area 1/8, by bisecting the right angle of each triangle. The second element or triangle of the tiling consists of any one, $G(j,2)$, of the $2^{j+1}-2$ subtriangles. Then subdivide the remaining $2^{j+1}-2-1$ into $2^{j+2}-4-2$ subtriangles each of area 1/16. Continue this procedure so that there are

Lemma 10:

$$G(j, j+k) = 2^{j+k-2} - 2^{k-2}G(j,1) - \cdots - 2G(j,k-2) - G(j,k-1) \tag{29}$$

subtriangles to be subdivided into triangles of area $1/2^{k+1}$, $k \geq 2$.

Proof: The proof follows by induction on j and k. For example,

$$\begin{aligned} 773 = G(4,12) = 2^{10} - 2^6 G(4,1) - 2^5 G(4,2) - 2^4 G(4,3) \\ - 2^3 G(4,4) - 2^2 G(4,5) - 2^1 G(4,6) - G(4,7). \end{aligned}$$

Hence, we see that the number of unchosen triangles is a Fibonacci j-sequence. From (29) and Lemma 5, we find that

$$\frac{G(j,j+k)}{2^{j+k-2}} = 1 - \sum_{i=1}^{k-1} \frac{G(j,i)}{2^{j+i-1}} \to 0, \ k \to +\infty,$$

which simply states that all of the area of the triangle $\bigcup_{i=0}^{j-2} T_i$ is tiled by this procedure.

This concept can be illustrated more formally in set-theoretic language. it is shown below that in the limit this procedure gives a tiled area equal to the area of the triangle $\bigcup_{i=0}^{j-2} T_i$. However, it is not clear that $\bigcup_{i=0}^{j-2} T_i$ is the union of all of these tiles. For example, when constructing the standard middle 1/3's Cantor set, the interval $[0,1]$ is not equal to the union of the middle 1/3's that are removed.

For given j, and $k \geq 1$, denote the set $\mathcal{F}_{j,k} = \{f(i, G(j,j)) : i = 1, \ldots, G(j,k)\}$ having as elements the $G(j,k)$ congruent subtriangles described by the k^{th} step of the procedure above. By construction, for given j, and each $k \geq 1$, the triangles $f(i, G(j,k))$ are pair-wise disjoint except for boundaries. Moreover, for each $j \geq 2$,

$$\begin{aligned} \text{area}\{\textstyle\bigcup_{k=1}^{+\infty} \mathcal{F}_{j,k}\} &= \text{area}\{\textstyle\bigcup_{k=1}^{+\infty} \bigcup_{i=1}^{G(j,k)} f(i, G(j,k))\} \\ &= \sum_{k=1}^{+\infty} \sum_{i=1}^{G(j,k)} \text{area}\{f(i, G(j,k))\} = 2^{j-2} = \sum_{k=1}^{+\infty} \frac{G(j,k)}{2^{k+1}}. \end{aligned}$$

This completes the proof of the theorem. It is observed that the theorem may be generalized by replacing $G(j,k)$ by an increasing sequence of positive integers $n(j,k)$ with the property that

$$\sum_{k=1}^{+\infty} \frac{n(j,k)}{2^{k+1}} = 2^{j-2}, \ 0 < n(j,k) < 2^{j+k-1}, \ k \geq 1.$$

For convenience in what follows, take the triangles $f(i, G(j,k))$ as open triangles, without boundary, thus interior$\big(f(i, G(j,k))\big) = f(i, G(j,k))$. Even though $\bigcup_{k=1}^{+\infty} \mathcal{F}_{j,k}$ is not necessarily the same set as $\bigcup_{i=0}^{j-2} T_i$, we have the following, where an overbar represents closure of a set.

Lemma 11: $\overline{\bigcup_{k=1}^{+\infty}\mathscr{F}_{j,k}} = \overline{\bigcup_{k=1}^{+\infty}\overline{\mathscr{F}_{j,k}}} = T_0 \cup T_1 \cup \cdots \cup T_{j-2} = \bigcup_{i=0}^{j-2} T_i$.

Proof: We denote $\bigcup_{i=0}^{j-2} T_i$ as right triangles previously defined together with the boundaries. Let $x \in \bigcup_{i=0}^{j-2} T_i$, then there is a sequence $\{x_k\} \in \bigcup_{k=1}^{+\infty}\mathscr{F}_{j,k}$ such that $x_k \neq x, |x-x_k| \to 0$ as $k \to +\infty$. Otherwise, there is $\varepsilon > 0$ such that

$$|x-x_\alpha| > \varepsilon,\ \forall x_\alpha \in \bigcup_{k=1}^{+\infty}\mathscr{F}_{j,k},$$

where α is the index of an uncountable set containing any possible sequence. Hence,

$$2^{j-2} = \text{area}\left\{\bigcup_{k=1}^{+\infty}\mathscr{F}_{j,k}\right\} \leq 2^{j-2} - \pi\varepsilon^2\delta,$$

where δ is the proportion of the ε-disk that intersects the triangle $T_0 \cup T_1 \cup \cdots \cup T_{j-2}$, a contradiction, so $x \in \overline{\bigcup_{k=1}^{+\infty}\mathscr{F}_{j,k}}$, i.e., x is an accumulation point of $\bigcup_{k=1}^{+\infty}\mathscr{F}_{j,k}$ (see Apostol [2]). Each x is arbitrary, which shows that $\bigcup_{i=0}^{j-2} T_i \subseteq \overline{\bigcup_{k=1}^{+\infty}\mathscr{F}_{j,k}}$. Equality holds, since the opposite set inclusion is true, that is, $\overline{f(i, G(j,k))} \subseteq \bigcup_{i=0}^{j-2} T_i, \forall i, k$. By a similar argument, $\overline{\bigcup_{k=1}^{+\infty}\overline{\mathscr{F}_{j,k}}} = \bigcup_{i=0}^{j-2} T_i$. The set $\bigcup_{i=0}^{j-2} T_i \setminus \bigcup_{k=1}^{+\infty}\mathscr{F}_{j,k}$ consists of points and straight line segments, i.e., contains the union of the boundary lines of the triangles $f(i, G(j,k))$ and $\bigcup_{i=0}^{j-2} T_i$. We note that $\bigcup_{i=0}^{j-2} T_i$ is closed and $\bigcup_{k=1}^{+\infty}\mathscr{F}_{j,k}$ is open, so that $\bigcup_{i=0}^{j-2} T_i \setminus \bigcup_{k=1}^{+\infty}\mathscr{F}_{j,k}$ is closed and bounded and, hence, compact.

Lemma 12: There is a "tiling" $\{f(i, G(2,k))\}$; $i = 1, \ldots, G(2,k)$; $k = 1, 2, \ldots$ of T_0 so that $E_2 \subseteq T_0 \setminus \bigcup_{k=1}^{+\infty}\mathscr{F}_{2,k}$, where the "tiling" has the same area as T_0 but is not necessarily equal to T_0.

Proof: To prove this, we let the open shaded triangles in the generation of E_2 be denoted by the triangles $f(i, G(2,k))$ which, by Theorem 5, tile T_0 and, hence, have the same area as T_0. To prove the first part of the lemma, it follows that if $x \in E_2$ then $x \notin f(i, G(2,k))$ for any i, k; thus, $x \notin \bigcup_{k=1}^{+\infty}\mathscr{F}_{2,k}$, and so $x \in T_0 \setminus \bigcup_{k=1}^{+\infty}\mathscr{F}_{2,k}$ since $x \in T_0$.

We note that $x \in E_2$ is not necessarily on a straight line segment or even a vertex of a triangle in $T_0 \setminus \bigcup_{k=1}^{+\infty}\mathscr{F}_{2,k}$. Analogously, the endpoints of the deleted intervals in the construction of the Cantor set are not the entire Cantor set.

Define $V\{T\}$ to be the set of vertices of a triangle denoted T. Then we find that

Theorem 6: $E_2 \supseteq \bigcup_{k=1}^{+\infty}\bigcup_{i=1}^{G(2,k)} V\{f(i, G(2,k))\}$.

Proof: That E_2 contains this set follows by the nature of the construction of E_2 and noting that two new vertices are added at level k, for each large unshaded triangle, which are the midpoint of the hypotenuse and the midpoint of an adjacent side. It is clear that no vertices are deleted once accumulated in E_2 by this process.

E_2 also has the property that

Corollary 2: Each point x in E_2 is the accumulation point of some countable sequence $\{x_k\}$ in E_2, such that $x_k \neq x$, $|x-x_k| \to 0$, $k \to +\infty$. That is, there are no isolated points in E_2.

Proof: Any vertex of $f(i, G(2, k))$ for some i, k is always a vertex of successively smaller triangles $f(i, G(2, k_0))$, $k_0 > k$, having vertices belonging to E_2 and limiting side lengths tending to 0 as $k \to +\infty$.

Example of a set E_j with dimension $2\log x_j / \log 2$

Define, for $j > 2$, the set $\widetilde{E}_j = \bigcup_{k=1}^{+\infty} \bigcup_{i=1}^{G(j,k)} V\{f(i, G(j, k))\}$ for an arbitrary tiling of $\bigcup_{i=0}^{j-2} T_i$. In words, this is the set of all possible vertices of the tiled triangles $f(i, G(j, k))$. This definition applies equally well to $j = 2$. For the sake of convenience, include $j = 2$ in the following analysis.

$\widetilde{E}_j$ is not necessarily closed, but is bounded. Let $\{x\}$ denote the set of accumulation points of $\widetilde{E}_j$ that lie in $\bigcup_{i=0}^{j-2} T_i \setminus \bigcup_{k=1}^{+\infty} \mathscr{F}_{j,k}$. Define $E_j = \widetilde{E}_j \cup \{x\}$ so that E_j contains all its accumulation points and is closed and bounded, so that $E_j \subseteq \Re^2$ as in Theorem 4.

To calculate the Box dimension, we observe that $(G(j, k))$ triangles $f(i, G(j, k))$, $i = 1, \ldots, G(j, k)$ are tiled at each level k. By induction, it can be shown that the number of ways to position the $G(j, k)$ unshaded triangles is $2G(j, k + j) = G(j, k) + G(j, k + j + 1)$, $k \geq 1$, as in Lemma 10 (except $k \geq 2$). We have

Theorem 7: $\forall j \geq 2$, E_j has Box dimension $= 2\log x_j / \log 2$.

Proof: For each k, the number of unshaded triangles forms twice a Fibonacci j-sequence, $2G(j, k + j)$, $\forall k \geq 1$ having hypotenuse length $1/\sqrt{2}^{k-1}$, each of which by Lemma 9 can be covered by two or three squares of side length $1/2\sqrt{2}^{k-1}$ or $1/2\sqrt{2}^{k}$, respectively. If we consider the latter, then at least one square is nonintersecting, except for boundaries, with other triangles. Hence, squares that overlap on different triangles cannot exceed $4G(j, k + j)$. Thus, it follows that the number of covering squares of size $1/2\sqrt{2}^{k}$ is at least $2G(j, k + j)$ and at most $6G(j, k + j)$, that is, the number of squares of side length $1/2\sqrt{2}^{k}$ that intersect E_j is bounded between the two scaled multiples of $G(j, k + j)$. By taking the limit as in Lemma 9 and applying a sandwich technique and Theorem 1, one obtains

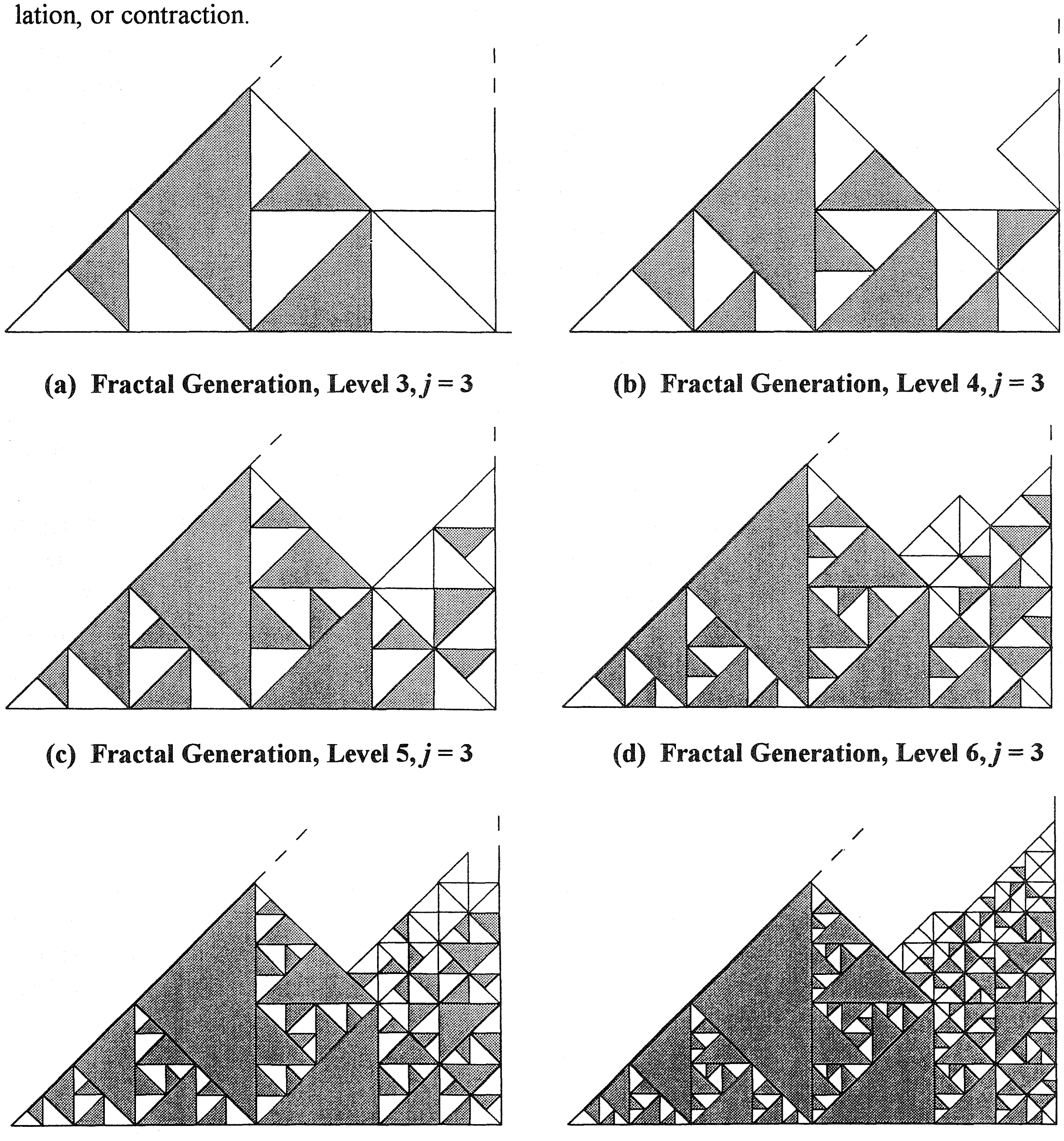
$$\frac{2\log x_j}{\log 2} = \lim_{k \to +\infty} \frac{\log 2G(j, k + j)}{\log 2\sqrt{2}^{k}} \leq D \leq \lim_{k \to +\infty} \frac{\log 6G(j, k + j)}{\log 2\sqrt{2}^{k}} = \frac{2\log x_j}{\log 2}.$$

This completes the proof of the theorem.

The construction of E_2 suggests that compact structures are formed by reflecting triangles of suitable size into an adjacent triangle. The affine transformations $\mathscr{T}, \mathscr{U}$ can be applied to form subsequent projections on the reflected triangles. E_j, $j \geq 3$, can be constructed with countably many copies of T_0, since the sequences $\{G(j, k)\}$ are the sum of countably many "shifted" sequences F_k. For example, for $j = 3$,

$$\begin{aligned}\{G(3, k)\} &= \{1, 1, 2, 4, 7, 13, 24, \ldots, G(3, k), \ldots\} \\ &= \{1, 1, 2, 3, 5, 8, 13, \ldots, F_k, \ldots\} + \{0, 0, 0, 1, 1, 2, 3, \ldots, F_k, \ldots\} \\ &\quad + \{0, 0, 0, 0, 1, 1, 2, \ldots, F_k, \ldots\} + \cdots + \{0, 0, \ldots, 0, 1, 1, 2, 3, 5, \ldots, F_k, \ldots\} + \cdots.\end{aligned}$$

Each of the sequences above corresponds to a scaled in size, tiled copy of T_0 which contains the fractal E_2 such that two copies of T_0 have the same area if the same number of zeros appear in the sequence. In the above, the sequences on the right-hand side correspond with triangles of area 1, 1/8, 1/16, ..., respectively, illustrated in Figures 6(a)-6(f), which show one of the possible ways to construct E_3, by reflecting unshaded triangles, projecting on these reflected triangles to create shaded triangles, and reflecting the new unshaded, smaller in area by 1/2, triangles so that their number forms a sequence $\{G(3, k)\}$. By Theorem 5, this process tiles $T_0 \cup T_1$ and generates E_3 with each point in E_3 on a translated, rotated, and contracted copy of E_2, and hence an accumulation point of E_3. We also note that the fractal dimension of E_2 is invariant under rotation, translation, or contraction.



(a) Fractal Generation, Level 3, $j = 3$

(b) Fractal Generation, Level 4, $j = 3$

(c) Fractal Generation, Level 5, $j = 3$

(d) Fractal Generation, Level 6, $j = 3$

(e) Fractal Generation, Level 7, $j = 3$

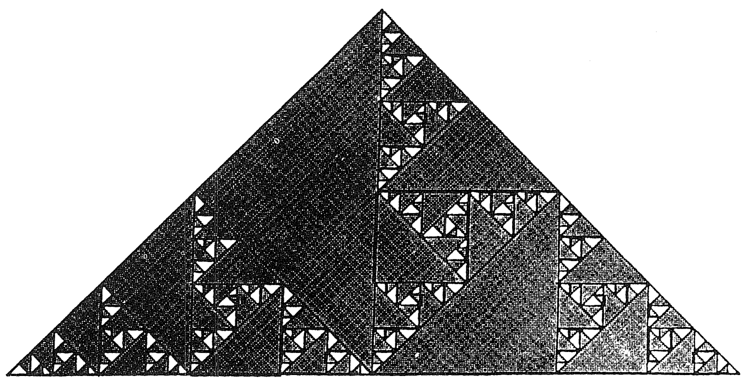
(f) Fractal Generation, Level 8, $j = 3$

FIGURE 6

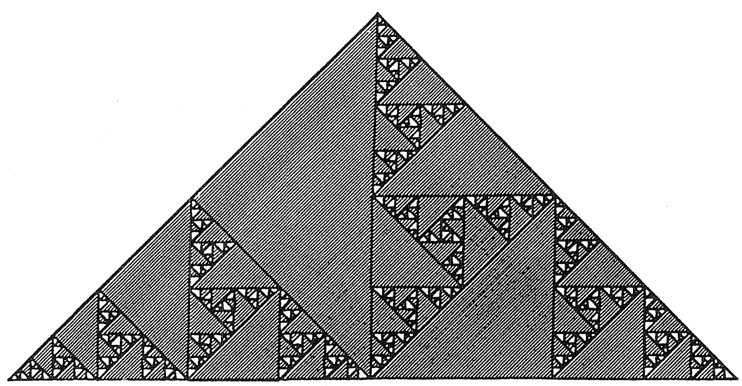
Figures 7(a) and 7(b) represent approximations of the fractal E_2 described in Section 3, and were constructed using *Logo* from a program by Robert G. Clason (see also [4]). The following dimensions were calculated on a *TI*–85©,

$$D_3 = 1.758292843,\ D_4 = 1.893554493,\ D_{10} = 1.998583839,\ D_{20} = 1.999998624.$$

As an interesting note, the projections on right triangles may be viewed as projections onto hyperplanes in $\mathfrak{R}^2$. This idea was also investigated in a more general setting in the manuscript of Angelos et al. [1].



(a) Fractal Generation, Level 10, *j* = 2



(b) Fractal Generation, Level 11, *j* = 2

FIGURE 7

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