A COMPOSITE OF MORGAN-VOYCE GENERALIZATIONS

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1. RATIONALE

Two recent papers, [1] and [3], detailed properties of

(i) a generalization $\{P_n^{(r)}(x)\}$ of the familiar Morgan-Voyce polynomials $B_n(x)$ and $b_n(x)$, and
(ii) an associated set $\{Q_n^r(x)\}$ of generalized polynomials.

Here, we amalgamate these two sets of polynomials into one more embracing class of polynomials $\{R_n^{(r,u)}(x)\}$.

In fact,

$$R_n^{(r,1)}(x) = P_n^{(r)}(x) \tag{1.1}$$

and

$$R_n^{(r,2)}(x) = Q_n^{(r)}(x). \tag{1.2}$$

Hopefully, the reader will have access to [1], [2], and [3]. However, the following summary may be helpful for reference purposes (in our notation):

$$P_n^{(0)}(x) = b_{n+1}(x), \tag{1.3}$$

$$P_n^{(1)}(x) = B_{n+1}(x), \tag{1.4}$$

$$P_n^{(2)}(x) = c_{n+1}(x), \tag{1.5}$$

$$Q_n^{(0)}(x) = C_n(x), \tag{1.6}$$

where $C_n(x)$ and $c_{n+1}(x)$ are polynomials related to the Morgan-Voyce polynomials. It may be mentioned that the polynomial $C_n(x)$ has already been defined by Swamy in [4], where it has been used in the analysis of Ladder networks. Knowledge of the definitions of the Fibonacci polynomials $\{F_n(x)\}$ and the Lucas polynomials $\{L_n(x)\}$ is assumed. When $x = 1$, the Fibonacci numbers F_n and the Lucas numbers L_n emerge.

Only the skeletal structure of the simple deductions from the definitions (2.1) and (2.2) germane to [1] and [3] will be displayed. This procedure follows the patterns in [1] and [3].

For internal consistency in my papers, I shall interpret symbolism in [1] in the notation adopted in [2] and [3]. Throughout, $n \geq 0$ except for the explicitly stated value $n = -1$.

Much of the material and approach offered in this paper appears to be new.

2. OUTLINE OF BASIC PROPERTIES OF $\{r_n^{(r,u)}(x)\}$

Definition

Define

$$R_n^{(r,u)}(x) = (x+2)R_{n-1}^{(r,u)}(x) - R_{n-2}^{(r,u)}(x) \quad (n \geq 2), \tag{2.1}$$

with

$$R_0^{(r,u)}(x) = u, \quad R_1^{(r,u)}(x) = x + r + u, \tag{2.2}$$

where r, u are integers. Then

$$R_n^{(r,u)}(x) = \sum_{k=0}^{n} c_{n,k}^{(r,u)} x^k , \tag{2.2}$$

with

$$c_{n,n}^{(r,u)} = 1 \text{ if } k \geq 1. \tag{2.4}$$

Recurrences

Clearly, from (2.1), $c_{n,0}^{(r,u)} = R_n^{(r,u)}(0)$ satisfies the recurrence

$$c_{n,0}^{(r,u)} = 2c_{n-1,0}^{(r,u)} - c_{n-2,0}^{(r,u)} \quad (n \geq 2), \tag{2.5}$$

with

$$\left.\begin{aligned} c_{0,0}^{(r,u)} &= u \\ c_{1,0}^{(r,u)} &= r + u \end{aligned}\right\}, \tag{2.6}$$

whence

$$c_{n,0}^{(r,u)} = nr + u, \tag{2.7}$$

$$\left.\begin{aligned} c_{n,0}^{(0,u)} &= u \\ c_{n,0}^{(r,0)} &= nr \end{aligned}\right\}, \tag{2.8}$$

$$\left.\begin{aligned} c_{n,0}^{(1,u)} &= n + u \\ c_{n,0}^{(r,1)} &= nr + 1 \end{aligned}\right\}. \tag{2.9}$$

Comparison of coefficients of x^k in (2.1) reveals the recurrence ($n \geq 2$, $k \geq 1$)

$$c_{n,k}^{(r,u)} = 2c_{n-1,k}^{(r,u)} - c_{n-2,k}^{(r,u)} + c_{n-1,k-1}^{(r,u)}. \tag{2.10}$$

The Coefficients $c_{n,k}^{(r,u)}$

Table 1 sets out some of the simplest of the coefficients $c_{n,k}^{(r,u)}$. For visual convenience in this table, we choose u to precede r.

From Table 1, [1], and [3], one may spot empirically the binomial formula

$$c_{n,k}^{(r,u)} = \binom{n+k-1}{2k-1} + r\binom{n+k}{2k+1} + u\binom{n+k-1}{2k} \tag{2.11}$$

$$= \binom{n+k}{2k} + r\binom{n+k}{2k+1} + (u-1)\binom{n+k-1}{2k}, \tag{2.12}$$

by Pascal's Theorem.

Multiply (2.12) throughout by x^k and sum. Accordingly,

Theorem 1: $R_n^{(r,u)}(x) = P_n^{(r)}(x) + (u-1)b_n(x)$.

Special cases:

$$R_n^{(0,1)}(x) = b_{n+1}(x) \quad \text{by (1.3), [2],} \tag{2.13}$$
$$R_n^{(1,1)}(x) = B_{n+1}(x) \quad \text{by (1.4), [2],} \tag{2.14}$$
$$R_n^{(2,1)}(x) = c_{n+1}(x) \quad \text{by (1.5), [2],} \tag{2.15}$$
$$R_n^{(0,2)}(x) = b_{n+1}(x) + b_n(x) = C_n(x) \quad \text{by [2].} \tag{2.16}$$

Furthermore,

$$R_n^{(0,0)}(x) = b_{n+1}(x) - b_n(x) = xB_n(x) \quad \text{by [2]}. \tag{2.17}$$

TABLE 1. The Coefficients $c_{n,k}^{(r,u)}$

$n \backslash k$	0	1	2	3	4	5	6
0	u						
1	$u+r$	1					
2	$u+2r$	$2+u+r$	1				
3	$u+3r$	$3+3u+4r$	$4+u+r$	1			
4	$u+4r$	$4+6u+10r$	$10+5u+6r$	$6+u+r$	1		
5	$u+5r$	$5+10u+20+r$	$20+15u+21r$	$21+7u+8r$	$8+u+r$	1	
6	$u+6r$	$6+15u+35r$	$35+35u+56r$	$56+28u+36r$	$36+9u+10r$	$10+u+r$	1
...	...	...	...	...	...	...	...

3. FIBONACCI AND LUCAS NUMBERS

Substitute $x = 1$ in Theorem 1. Then, with $F_n(1) = F_n$ and $L_n(1) = L_n$,

$$\begin{aligned} R_n^{(r,u)}(1) &= F_{2n+1} - F_{2n-1} + rF_{2n} + uF_{2n-1} \\ &= (1+r)F_{2n} + uF_{2n-1}. \end{aligned} \tag{3.1}$$

For example, $R_4^{(r,u)}(1) = 21 + 21r + 13u = (1+r)F_8 + uF_7$, as may be verified quickly in Table 1.

Special cases:

$$R_n^{(0,1)}(1) = F_{2n+1}, \tag{3.2}$$

$$R_n^{(1,1)}(1) = F_{2n+2}, \tag{3.3}$$

$$R_n^{(2,1)}(1) = L_{2n+1}, \tag{3.4}$$

$$R_n^{(0,2)}(1) = L_{2n}. \tag{3.5}$$

Also,

$$R_n^{(0,0)}(1) = F_{2n}. \tag{3.6}$$

Relationships between the Fibonacci and Lucas numbers, and the Morgan-Voyce polynomials when $x = 1$, are specified in [2].

4. CHEBYSHEV POLYNOMIALS

Write

$$\frac{x+2}{2} = \cos t \quad (-4 < t < 0). \tag{4.1}$$

In [2], it is shown that

$$B_n(x) = U_n\left(\frac{x+2}{2}\right), \tag{4.2}$$

$$b_n(x) = U_n\left(\frac{x+2}{2}\right) - U_{n-1}\left(\frac{x+2}{2}\right), \tag{4.3}$$

$$c_n(x) = U_n\left(\frac{x+2}{2}\right) + U_{n-1}\left(\frac{x+2}{2}\right), \tag{4.4}$$

$$C_n(x) = 2T_n\left(\frac{x+2}{2}\right), \tag{4.5}$$

where $U_n(x)$ and $T_n(x)$ are Chebyshev polynomials.

Empirically, (4.2)-(4.5), taken with (2.13)-(2.16), suggest a more general formula connecting $R_n^{(r,u)}(x)$ with the Chebyshev polynomials.

Theorem 2: $R_n^{(r,u)}(x) = U_{n+1}\left(\frac{x+2}{2}\right) + (r+u-2)U_n\left(\frac{x+2}{2}\right) - (u-1)U_n\left(\frac{x+2}{2}\right).$

Thus, in particular

$$R_n^{(r,1)}(x) = U_{n+1}\left(\frac{x+2}{2}\right) + (r-1)U_n\left(\frac{x+2}{2}\right) \tag{4.6}$$

and

$$R_n^{(r,2)}(x) = 2T_n\left(\frac{x+2}{2}\right) + rU_n\left(\frac{x+2}{2}\right). \tag{4.7}$$

Zeros and orthogonality properties of $B_n(x)$, $b_n(x)$, $c_n(x)$, and $C_n(x)$ may be found in [5], [4], and [2]. *En passant*, the zeros of $R_3^{(0,0)}(x)$, say, are, by (2.17), the zeros of $xB_3(x)$, namely, $0, -1, -2$.

5. THREE IMPORTANT PROPERTIES

Roots $\alpha(x) = \alpha$ and $\beta(x) = \beta$ of the characteristic equation for (2.1), namely,

$$\lambda^2 - (x+2)\lambda + 1 = 0, \tag{5.1}$$

are

$$\begin{cases} \alpha = \dfrac{x+2+\sqrt{x^2+4}}{2}, \\ \beta = \dfrac{x+2-\sqrt{x^2+4}}{2}, \end{cases} \tag{5.2}$$

whence

$$\begin{cases} \alpha\beta = 1, \\ \alpha+\beta = x+2, \\ \alpha-\beta = \sqrt{x^2+4x}. \end{cases} \tag{5.3}$$

The Binet form for $B_n(x)$ is, by [2],

$$B_n(x) = \frac{\alpha^n - \beta^n}{\alpha - \beta}. \tag{5.4}$$

Moreover, by [2],

$$(x+1)B_n(x) - B_{n-1}(x) = b_{n+1}(x), \tag{5.5}$$

$$(x+2)B_n(x) - B_{n-1}(x) = B_{n+1}(x), \tag{5.6}$$

$$(x+3)B_n(x) - B_{n-1}(x) = c_{n+1}(x), \tag{5.7}$$

$$(x+2)B_n(x) - 2B_{n-1}(x) = C_n(x). \tag{5.8}$$

Standard methods involving (2.1) and (2.2) yield the *Binet form* for $R_n^{(r,u)}(x)$.

Theorem 3: $R_n^{(r,u)}(x) = \dfrac{(x+r+u)(\alpha^n-\beta^n)-u(\alpha^{n-1}-\beta^{n-1})}{\alpha-\beta}$

$= (x+r+u)B_n(x) - uB_{n-1}(x)$, by (5.4).

Use of Theorem 3 in conjunction with (5.5)-(5.8) returns us to (2.13)-(2.16). Next, we record that, from (2.1) and (2.2),

$$R_{-1}^{(r,u)} = (u-1)x + u - r, \tag{5.9}$$

whence, by (2.13)-(2.16), $B_0(x) = 0$, $b_0(x) = 1$, $c_0(x) = -1$, and $C_{-1}(x) = x+2$.

Successive applications of the Binet form (Theorem 3) eventually give, on simplification and use of (2.2), (5.4), and (5.9), the *Simson formula*

Theorem 4: $\left.\begin{aligned} R_{n+1}^{(r,u)}(x)R_{n-1}^{(r,u)}(x) - [R_n^{(r,u)}(x)]^2 &= (x+r+u)[(u-1)x+u-r]-u^2 \\ &= R_1^{(r,u)}(x)R_{-1}^{(r,u)}(x) - [R_0^{(r,u)}(x)]^2. \end{aligned}\right\}$

Familiar techniques produce the *generating function* (Theorem 5) to complete our trilogy of salient features of $R_n^{(r,u)}(x)$.

Theorem 5: $\left.\begin{aligned} \sum_{i=0}^{\infty} R_i^{(r,u)}(x)y^i &= \frac{u-\{(u-1)x+u-r\}y}{1-(x+2)y+y^2} \\ &= \frac{R_0^{(r,u)}(x) - R_{-1}^{(r,u)}(x)y}{1-(x+2)y+y^2} \quad \text{by (2.2), (5.9).} \end{aligned}\right\}$

Special cases of Theorems 4 and 5:

r	u	$R_n^{(r,u)}(x)$	R.H.S. of Th. 4	Numerator in Th. 5
0	1	$b_{n+1}(x)$	x	$1-y$
1	1	$B_{n+1}(x)$	-1	1
2	1	$c_{n+1}(x)$	$-(x+4)$	$1+y$
0	2	$C_n(x)$	$x(x+4)$	$2-(2+x)y$

Observe that, in the third column, (i) row 1 × row 3 = row 2 × row 4, (ii) row 3 = $-\frac{(\alpha-\beta)^2}{x}$, (iii) row 4 = $(\alpha-\beta)^2$.

6. RISING DIAGONAL FUNCTIONS

Imagine, in the mind's eye, a set of parallel upward-slanting diagonal lines in Table 1 that delineate the *rising diagonal functions* $\mathfrak{R}_n^{(r,u)}(x)$ [$= \mathfrak{R}_n(x)$ for brevity] defined by

$$\mathfrak{R}_n(x) = \sum_{k=0}^{\left[\frac{n+1}{2}\right]} c_{n+1-k,k}^{(r,u)}(x)x^k \tag{6.1}$$

with

$$\mathfrak{R}_0(x) = r+u, \ \mathfrak{R}_1(x) = x+2r+u, \tag{6.2}$$

where the values of the coefficients of x^k in (6.1) are given in (2.5)-(2.12).

Thus, for example,

$$\mathcal{R}_4(x) = c_{5,0}^{(r,u)} + c_{4,1}^{(r,u)}x + c_{3,2}^{(r,u)}x^2 = (5r+u) + (4+10r+6u)x + (4+r+u)x^2$$

as may be checked in Table 1.

Choosing $\mathcal{R}_0(x) = r+u$ involves a slightly subtle point. If one allows negative subscripts of $\mathcal{R}_n(x)$, then the diagonal function $\mathcal{R}_{-1}(x)$ is not equal merely to u, but to a more complicated expression.

Some intriguing results of a fundamental nature for $\{\mathcal{R}_n^{(r,u)}(x)\}$ now emerge. First, we discover the recurrence relation. For this we need, by (2.11),

$$c_{\frac{n}{2}+1,\frac{n}{2}}^{(r,u)} = n+r+u \quad (n \text{ even}). \tag{6.3}$$

Theorem 6: $\mathcal{R}_n(x) = 2\mathcal{R}_{n-1}(x) + (x-1)\mathcal{R}_{n-2}(x) \quad (n \geq 2)$.

Proof: Use (6.1). Sum for each power of x for $k = 0, 1, 2, \ldots, [\frac{n-1}{2}]$ and simplify according to (2.4), (2.7), (2.10), (2.11), and (6.3). Then

$$\begin{aligned} 2\mathcal{R}_{n-1}(x) + (x-1)\mathcal{R}_{n-2}(x) &= 2\sum_{k=0}^{[\frac{n}{2}]} c_{n-k,k,}x^k - \sum_{k=0}^{[\frac{n-1}{2}]} c_{n-1-k,k}x^k + \sum_{k=0}^{[\frac{n-1}{2}]} c_{n-1-k,k}x^{k+1} \\ &= c_{n+1,0} + c_{n,1}x + \cdots + c_{n+1-m,m}x^m + \cdots + \begin{cases} n+r+u, & n \text{ even}, \\ 1 & n \text{ odd}, \end{cases} \\ &= \mathcal{R}_n(x). \end{aligned}$$

Corollary 1: $\mathcal{R}_n(1) = 2^{n-1}(1+2r+u)$.

Proof:

$$\begin{aligned} \mathcal{R}_n(1) &= 2\mathcal{R}_{n-1}(1) && \text{by Th. 6,} \\ &= 2^2\mathcal{R}_{n-2}(1) && \text{by Th. 6 again,} \\ &\cdots \\ &= 2^{n-1}\mathcal{R}_1(1) && \text{by repeated use of Th. 6,} \\ &= 2^{n-1}(1+2r+u) && \text{by (6.2).} \end{aligned}$$

Special cases: Substituting in Corollary 1 the values of r and u appropriate to $B_n(x)$, $b_n(x)$, $c_n(x)$, and $C_n(x)$, we obtain the corresponding values for the diagonal functions of these polynomials when $x = 1$, as stated in the concluding segment of [2].

From Theorem 6, the characteristic equation for $\mathcal{R}_n^{(r,u)}(x)$ is $\lambda^2 - 2\lambda - (x-1) = 0$ with roots $\gamma(x) = \gamma$, $\delta(x) = \delta$ expressed by

$$\begin{cases} \gamma = 1+\sqrt{x}, \\ \delta = 1-\sqrt{x}, \end{cases} \tag{6.4}$$

so that

$$\begin{cases} \gamma+\delta = 2, \\ \gamma\delta = 1-x, \\ \gamma-\delta = 2\sqrt{x}. \end{cases} \tag{6.5}$$

In the standard process for the derivation of the generating function of $\mathcal{R}_n(x)$, a fine nuance presents itself, namely, the recognition that, by (6.2),

$$\mathcal{R}_3(x) - 2\mathcal{R}_2(x) = x+2r+u-2(r+u) = x-u. \tag{6.6}$$

Applying Theorem 6 and (6.4), our treatment creates the following generating function.

Theorem 7: $\sum_{i=0}^{\infty} \mathcal{R}_i(x)y^i = \{r+u+(x-u)y\}[1-(2y+(x-1)y^2)]^{-1}$.

Straightforward techniques yield the Binet form

Theorem 8: $\mathcal{R}_n(x) = \dfrac{\{\mathcal{R}_1(x)-\delta\mathcal{R}_0(x)\}\gamma^n - \{\mathcal{R}_1(x)-\gamma\mathcal{R}_0(x)\}\delta^n}{\gamma-\delta}$.

Finally, by Theorem 8, we derive the Simson formula

Theorem 9: $\mathcal{R}_{n+1}(x)\mathcal{R}_{n-1}(x) - \mathcal{R}_n^2(x) = (-1)^n(x-1)^{n-1}\{(r+x)^2 - x(r+u)^2\}$.

It is clear from Theorem 9, or from Corollary 1, that

$$\mathcal{R}_{n+1}(1)\mathcal{R}_{n-1}(1) = \mathcal{R}_n^2(1). \tag{6.7}$$

The particular situations for $B_n(x)$, $b_n(x)$, $c_n(x)$, and $C_n(x)$ in relation to Theorems 6-9 may be readily deduced.

7. CONCLUDING THOUGHTS

There does seem to be scope for further developments. One such advance, for instance, might be the extension of the theory through negative subscripts of $\mathcal{R}_n^{(r,u)}(x)$. Recall (5.9) for $n=-1$.

Another innovation is the consideration of the replacement of $x+2$ by $x+k$ (k integer). And what of interest might eventuate if $k=r$? $k=u$?

Possibly, some worthwhile differential equations could be hidden among the $\mathcal{R}_n^{(r,u)}(x)$. Experience teaches us that this is often the case when exploring diagonal functions.

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