DYNAMICS OF THE MÖBIUS MAPPING AND FIBONACCI-LIKE SEQUENCES

I. Jaroszewski and A. K. Kwaśniewski*
Institute of Physics, Warsaw University, Campus Białystok
Computer Science Laboratory ul. Przytorowa 2a
Pl-15-104 Bialystok, Poland
*e-mail: kwandr@cksr.ac.białystok.pl
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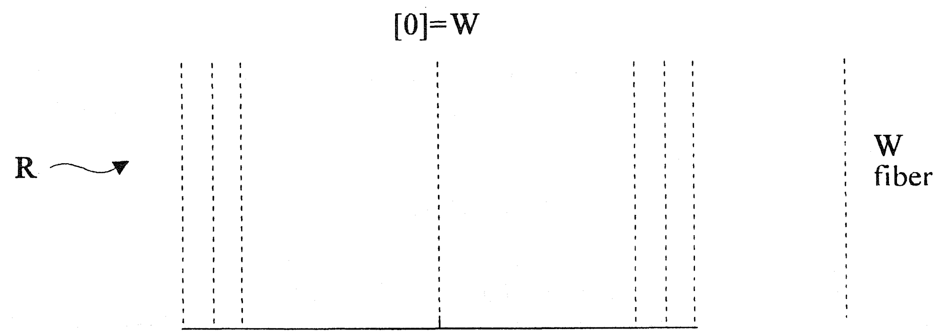
Any two numbers $\varsigma, \eta \in \mathbf{R}$ are equivalent ($\varsigma \sim \eta$) if and only if there exists

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U(2, \mathbf{Z}) \equiv \{A \in M_2(\mathbf{Z}); |\det A| = 1\},$$

such that

$$\varsigma = f_A(\eta) \equiv \frac{a\eta + b}{c\eta + d}.$$

It is well known [4] that the above equivalence relation "$\sim$" provides us with the following fibration of $\mathbf{R}$:



B = Base = irrationals $\cup$ {0}

Consider now the dynamical system $(\mathbf{R}, f_A)$ with the specially chosen Möbius mapping $f_A : \mathbf{R} \to \mathbf{R}$; $A \in U(2, \mathbf{Z})$. One sees then that f_A acts along fibers. That is,

$$\forall b \in B : [b] \ni x \to f_A(x) \in [b] \Rightarrow \forall n \in \mathbf{N} : \left\{f_A^k(x)\right\}_{k=1}^{n} \subset [b].$$

(Naturally, $f_A^n = f_{A^n}$.)

An example of such dynamics is $(\mathbf{R}, f_A)$ with $A = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right) \in U(2, \mathbf{Z})$. This was investigated in [3].

In this note, the authors give a concise presentation of the dynamics generated by iteration of the arbitrary Möbius transformation $f_{\hat{A}}$; $\hat{A} = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$; $\det \hat{A} \equiv -t \neq 0$.

In view of the Cayley-Hamilton theorem, it is enough to consider the matrices of the form $A = \left(\begin{smallmatrix} s & t \\ 1 & 0 \end{smallmatrix}\right)$, where $s = \operatorname{Tr} \hat{A}$ and $t = -\det \hat{A}$; $\hat{A} \in \mathrm{GL}(2, \mathbf{R})$.

Naturally,

$$\hat{A}^2 = s\hat{A} + t\mathbf{1};\ \mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence

$$\hat{A}^{n+1} = H_{n+1}\hat{A} + tH_n\mathbf{1}, \tag{1}$$

where

$$H_{n+2} = sH_{n+1} + tH_n,\ H_0 = 0, H_1 = 1;\ n \in \mathbf{N} \cup \{0\} \tag{2}$$

and

$$H_n(s,t) \equiv H_n.$$

It is also easy to see that when $A = \left(\begin{smallmatrix} s & t \\ 1 & 0 \end{smallmatrix}\right)$,

$$A^n = \begin{pmatrix} H_{n+1} & tH_n \\ H_n & tH_{n-1} \end{pmatrix};\ n \in \mathbf{N}. \tag{3}$$

The singular point of the transformation f_A is 0. However, this point is never reached unless one chooses $x_0 \in S_A$ (or $x_0 = 0$) as a starting point, where

$$S_A = \left\{v_n \in \mathbf{R};\ v_n = f_A^{-n}(0);\ n \in \mathbf{N}\right\} \Rightarrow S_A = \left\{v_n;\ v_n = -t\frac{H_n}{H_{n+1}};\ n \in \mathbf{N}\right\}.$$

Note, however, that for $A \notin U(2,\mathbf{Z})$ the trajectories $\{f_A^n(x);\ x \notin S_A;\ n \in \mathbf{N}\}$ run across $[b] \sim \mathbf{W}$ fibers of $\mathbf{R}$.

It is also useful to note the following. Let us call $(\mathbf{R}, f_{\hat{A}})$ and $(\mathbf{R}, f_{\hat{B}})$ equivalent and write $(\mathbf{R}, f_{\hat{A}}) \sim (\mathbf{R}, f_{\hat{B}})$ if and only if $\exists U \in \mathrm{GL}(2,\mathbf{R});\ \hat{B} = U^{-1}\hat{A}U$. Then the characteristic points of the dynamical system, that is, the set $S_{\hat{B}}$ (see the definition of S_A), the attracting (stable) fixed point as well as the unstable fixed point of the $(\mathbf{R}, f_{\hat{B}})$ system are just the corresponding characteristic points of $(\mathbf{R}, f_{\hat{A}})$ shifted by f_U Möbius transformation. For example,

$$\left(\mathbf{R}, f_{\left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)}\right) \text{ of [3] is equivalent to } \left(\mathbf{R}, f_{\left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)}\right) \text{ with } U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

As far as these characteristic points of the dynamic system $(\mathbf{R}, f_{\hat{A}})$ are concerned, the overall picture of all dynamics is the same as in [3] under the condition that there are two fixed points of f_A, that is, we have

$$x_\pm^* = \frac{s \pm \sqrt{s^2 + 4t}}{2} \text{ where } s^2 + 4t > 0 \tag{4}$$

and

$$\left|\frac{d}{dx} f_A(x)\right|_{x = x_+^*} < 1, \tag{5}$$

$$\left|\frac{d}{dx} f_A(x)\right|_{x = x_-^*} > 1. \tag{6}$$

Conditions (5) and (6) impose calculable restrictions on the s and t parameters. If these are satisfied, then x_+^* is a stable attracting point. That is, the sequence $x_n = f_A^n(x_0)$, $x_0 \notin S_A$ converges to x_+^* (almost regardless of the choice of starting point x_0). The x_-^* is then the unstable fixed point. When $x_0 \neq x_-^*$, the sequence x_n converges to x_-^* if and only if $\exists N;\ \forall n > N;\ x_n = x_-^*$. One proves this via *a contratio* reasoning (see [2]). Explicitly, one has, for any unstable fixed point

$$\forall x_0 \in \mathbf{U}_A;\ x_n \to x_-^*,$$

where

$$\mathbf{U}_A = \left\{\chi_n;\ \chi_n = f_A^{-n}(x_-^*)\ n \in \mathbf{N}\right\} \Rightarrow \mathbf{U}_A = \left\{\chi_n;\ \chi_n = t\frac{\Xi_n}{\Xi_{n+1}}\ n \in \mathbf{N}\right\}, \tag{7}$$

where

$$\Xi_{n=2} = s\cdot\Xi_{n+1} - \Xi_n,\ \Xi_0 = -x_-^*,\ \Xi_1 = 1. \tag{8}$$

That is, apart from the set S_A another characteristic set $\mathbf{U}_A$ is attributed to the dynamical system$(\mathbf{R}, f_A)$.

However, conditions (5) and (6) need not be met. For example, $f_A;\ A = \left(\begin{smallmatrix}2 & -1\\ 1 & 0\end{smallmatrix}\right)$ has only one fixed point $x_+^* = x_-^* \equiv x^*$, and

$$\left|\frac{d}{dx}f_A(x^*)\right| = 1.$$

It is easy to see that, for all x_0, $x_0 \notin S_A$, $f_A^n(x_0) \xrightarrow{n\to\infty} 1$.

However, starting at $x_0 = 1-\varepsilon$ ($\varepsilon > 0$; ε small) the iterates x_n move away from 1. Hence, x^* is not an attracting fixed point. Note the difference from (6); the argument giving rise to the set $\mathbf{U}_A$ necessitates an inequality $\left|\frac{d}{dx}f_A(x^*)\right| > 1$ for an unstable fixed point (see [2]).

Following [2], one states that, to any single fixed point $x^* = x_+^* = x_-^*$, there corresponds a set $\{f_A;\ \beta \neq 0\}$ of Möbius maps where

$$A = \begin{pmatrix} x^* & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} 1 & \beta\\ 0 & 1\end{pmatrix}\begin{pmatrix} 0 & 1\\ 1 & -x^*\end{pmatrix};\ \beta \neq 0.$$

Since $s = \operatorname{Tr} A = 2$ and $t = -\det A = -1$, the above f_A Möbius transformation with $A = \left(\begin{smallmatrix}2 & -1\\ 1 & 0\end{smallmatrix}\right)$ is representative of the whole class of equivalent dynamical systems $\{(\mathbf{R}, f_{\hat{A}});\ \operatorname{Tr}\hat{A} = 2, \det\hat{A} = 1\}$. (Note that f_A acts along $\mathbf{W}$-fibers of $\mathbf{R}$.)

In conclusion, we state that the general features of

$$\left(\mathbf{R}, f_{\left(\begin{smallmatrix}0 & 1\\ 1 & 1\end{smallmatrix}\right)}\right)$$

dynamics described in [3] are typical for the dynamics $(\mathbf{R}, f_A)$ when $s^2+4t > 0$, where $\operatorname{Tr} A = s$, $\det A = -t$. For $s^2+4t > 0$, one has one stable attracting point x_+^* and one unstable repelling point x_-^*, as then

$$A \sim \begin{pmatrix} x_+^* & 0 \\ 0 & x_-^* \end{pmatrix} \equiv D.$$

That is, $(\mathbf{R}, f_A) \sim (\mathbf{R}, f_D)$ and $f_U(0) = x_-^*$, while $f_U(\infty) = x_+^*$;

$$U = \begin{pmatrix} 1 & 1 \\ 1/x_+^* & 1/x_-^* \end{pmatrix}.$$

The fixed point x_-^* is therefore the repelling one even for

$$\left|\frac{d}{dx} f_A(x_-^*)\right| = 1 \Leftrightarrow \{s^2 + 4t = 1 \vee 2s^2 = 1 + \sqrt{1+16t}\}.$$

The general features of the $(\mathbf{R}, f_A)$ dynamical system apart from fixed points consist of two descending sequences of intervals

$$\{[\mu_n, \mu_{n+1}]\};\ \mu_n = \frac{H_{n+1}}{H_n};\ \text{ and }\ \{[\nu_n, \nu_{n+1}]\};\ \nu_n = -t\frac{H_n}{H_{n+1}};$$

which, by virtue of (3), converge correspondingly to x_+^* and $x_-^* \equiv -t/x_+^*$.

In this note, we also notice that $\mathbf{U}_A$, the set of points defined by (7) and (8), is attributed to the $(\mathbf{R}, f_A)$ dynamical system with an unstable fixed point x_-^*.

The detailed behavior of the $(\mathbf{R}, f_A)$ iterative system is then finally established by the following sequence of bijections (for $t > 0$, $s > 0$):

$$\begin{aligned}
&f_A\colon (\nu_2, 0) \to (-\infty, \nu_1),\\
&f_A\colon (-\infty, \nu_1) \to (0, \infty),\\
&f_A\colon (\nu_{2n+2}, \nu_{2n}) \to (\nu_{2n-1}, \nu_{2n+1}),\\
&f_A\colon (\nu_{2n+1}, \nu_{2n+3}) \to (\nu_{2n+2}, \nu_{2n}),\\
&f_A\colon (0, x_+^*] \to [x_+^*, \infty),\\
&f_A\colon [x_+^*, \infty) \to (0, x_+^*].
\end{aligned}$$

The above shows that any point $x_0 \in \mathbf{R}$ (such that $x_0 \notin \mathbf{U}_A$ and $x_0 \notin S_A$) escapes from any vicinity of x_-^* and runs to x_+^*. This is also illustrated in the figures presented below.

The case of $s^2 + 4t = 0$ is the limit case. Thus, one has

$$x^* = x_+^* = x_-^* = \frac{s}{2};\ \mu_n = \frac{s}{2}\left(1 + \frac{1}{n}\right) \to x^*$$

and

$$S_A = \left\{\nu_n = -t\frac{H_n}{H_{n+1}} = \frac{s}{2}\cdot\frac{n}{n+1};\ n \in \mathbf{N}\right\}$$

because the Fibonacci-like sequence $\{H_n\}$ is now given by $H_n = (s/2)^{n-1}\cdot n$; $n \in \mathbf{N}$, and $H_0 = 0$.

As in the case of $A = \left(\begin{smallmatrix} 2 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ considered above, we have for all $x_0 \in \mathbf{R}$; $x_0 \notin S_A \cup \{0\}$ ($s^2 + 4t = 0$):

$$f_A^n(x_0) \xrightarrow{n\to\infty} \frac{s}{2}.$$

One also easily sees from

$$f_A^n(x^* + \varepsilon) = x^* \frac{x^* + (n+1)\varepsilon}{x^* + n\varepsilon}$$

that, for small ε, the first iterates $x_n \equiv f_A^n(x^* + \varepsilon)$ are attracted or repelled, depending on whether x^* and ε are of the same sign or not. The fixed point $s/2$ is therefore neither attracting nor repelling.

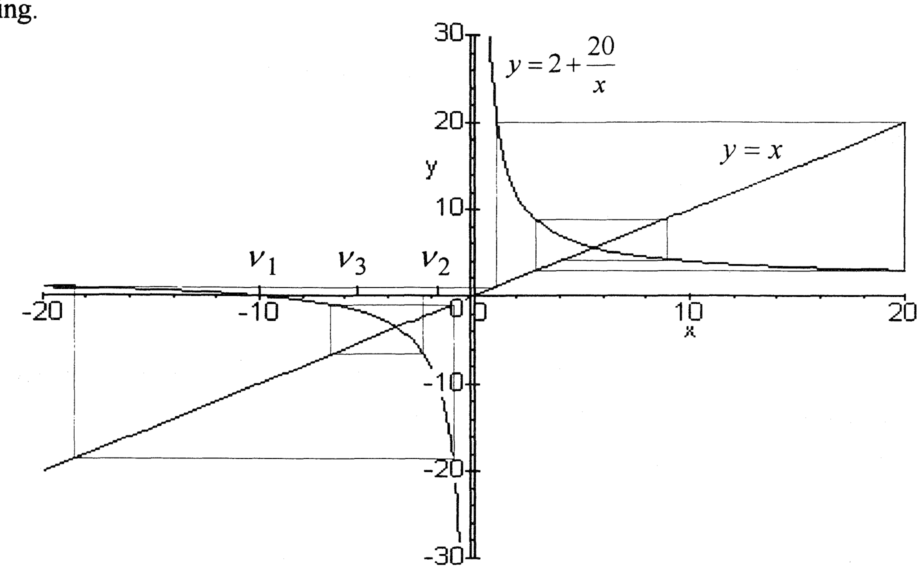


FIGURE 1. Illustration of the General Behavior of the Dynamical System with Two Fixed points ($s = 1$; $t = 20$)

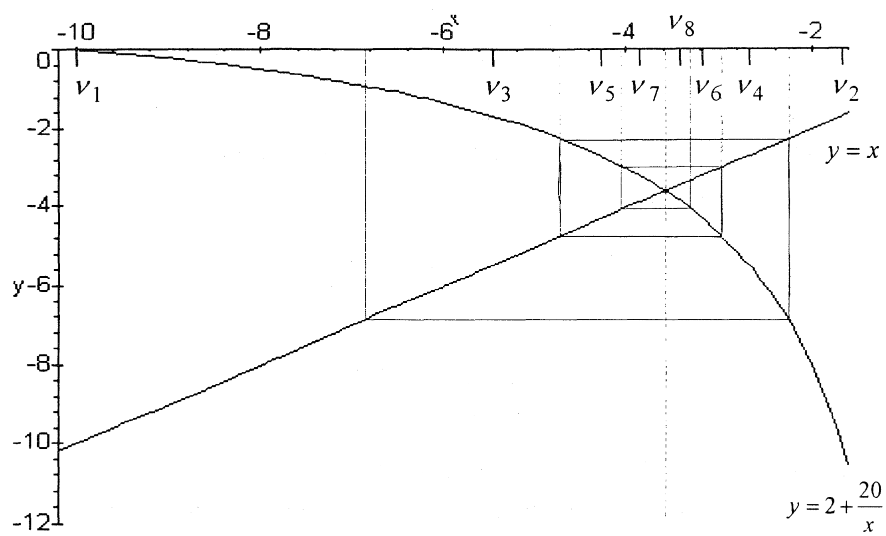


FIGURE 2. Magnification of the x_-^* Neighborhood from Figure 1

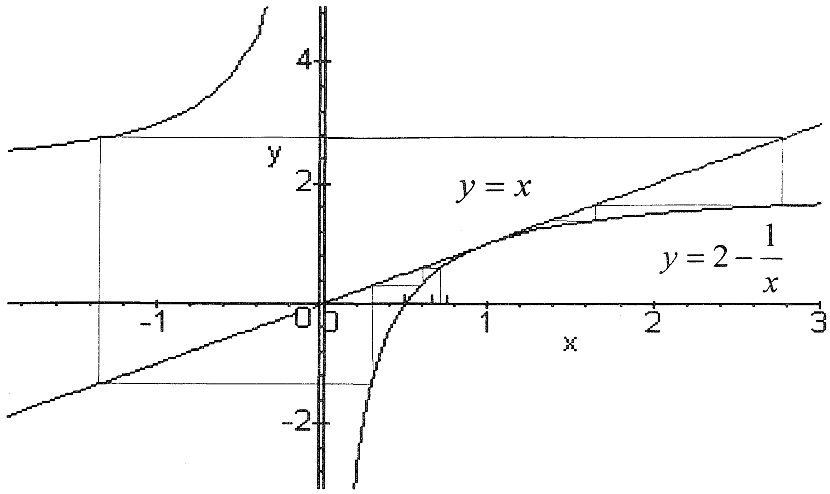


FIGURE 3. Illustration of the General Behavior of the Dynamical System with One Fixed Point ($s = 2$; $t = -1$)

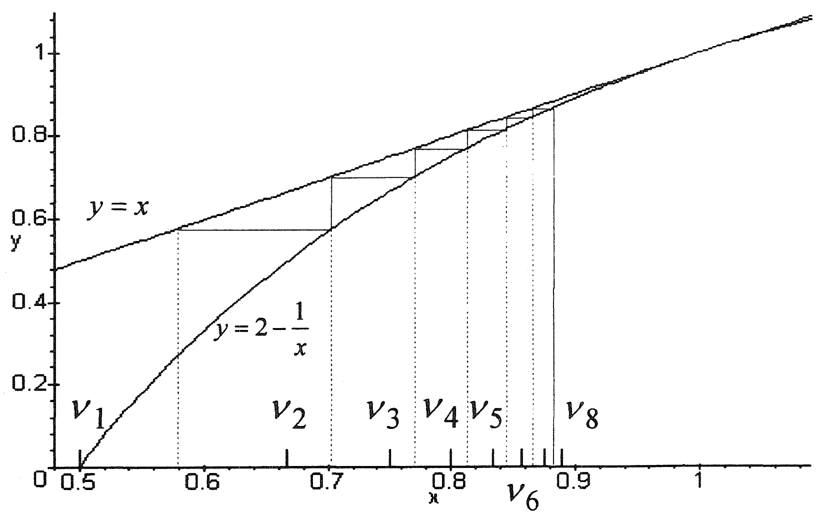


FIGURE 4. Magnification of the x_-^* Neighborhood from Figure 3

In the case of $s^2 + 4t = 0$, $s > 0$, the detailed behavior of the $(\mathbf{R}, f_A)$ iterative system is established again through the following sequence of bijections:

$$\begin{aligned} &f_A: (s/2, \infty) \to (s/2, \infty), \\ &f_A: (-\infty, 0) \to (s/2, \infty), \\ &f_A: (v_1, 0) \to (-\infty, 0), \\ &f_A: (v_1, v_2) \to (0, v_1), \\ &f_A: (v_{n+1}, v_{n+2}) \to (v_n, v_{n+1}). \end{aligned}$$

The cases $s = 0$ and $s^2 + 4t < 0$ (that is, without *real* fixed points) are easily treated, too (see [2]). In this case, one may encounter also finite periodic orbits (as, for example,

$$\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}^6 = 1 \text{ or } \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}^3 = 1,$$

etc.) if

$$\exists n \in N; \left(\frac{x_+^*}{x_-^*}\right)^n = 1;$$

otherwise, orbit forms a dense subset of an interval.

The presented investigation also provides one with some general insights that are useful for describing the $(\mathscr{C}, f_A)$ dynamical system, where $\mathscr{C}$ stands for Clifford algebra and f_A is a corresponding Möbius transformation in $\mathbf{R}^n$ (see [1]). There, the Clifford numbers' valued Fibonacci-like sequences play a role similar to that of the $\{H_n\}_0^\infty$ and $\{\Xi_n\}$ sequences in the $\mathbf{R}$ case.

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