THE GOLDEN STAIRCASE AND THE GOLDEN LINE

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(Submitted May 1993–Final Revision February 1997)

0. INTRODUCTION

An arrangement of squares dissected from the golden rectangle and placed along the positive X-axis creates a golden staircase, upon which sits the golden line. In this paper we consider some algebraic and geometric relationships expressed in this figure. An extended figure depicts infinite series and relationships involving Fibonacci numbers.

1. THE GOLDEN LINE

In a golden rectangle, the ratio of the length to the width is the same as the ratio of the sum of the length and the width to the length. This golden ratio ϕ has a value of $(1+\sqrt{5})/2$. Consider a golden rectangle with length ϕ and width 1. When a square with side 1 is inscribed as in Figure 1, the remaining rectangle has a length to width ratio of $1/(\phi-1)$, which simplifies to ϕ, establishing a nice relationship between ϕ and its reciprocal

$$\phi-1=\frac{1}{\phi}, \tag{1}$$

which is equivalent to $\phi^2=\phi+1$. When continued, this partitioning process generates the familiar infinite progression of spiraling squares, the first few of which are shown in Figure 2.

When all the squares are placed along the positive X-axis creating the golden staircase shown in Figure 3, the upper right corners of the squares are collinear and define the golden line, which has equation $y=(-1/\phi)x+\phi$. The equation of the line AB through the upper right corners of the first two squares is $y-1=\frac{(1/\phi-1)}{(\phi-1)}(x-1)$, which simplifies to

$$y=\frac{-1}{\phi}x+\phi. \tag{2}$$

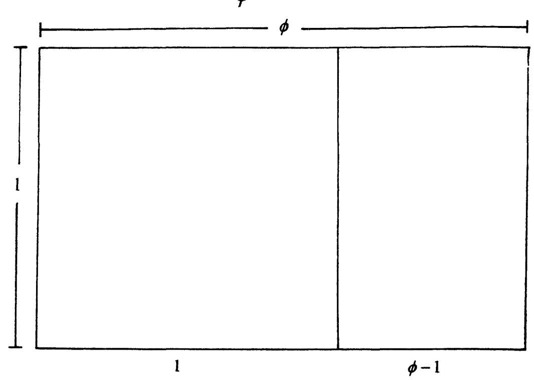

FIGURE 1

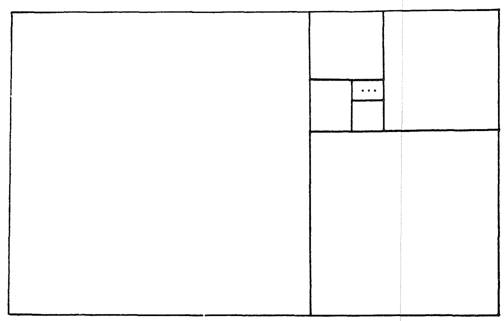

FIGURE 2

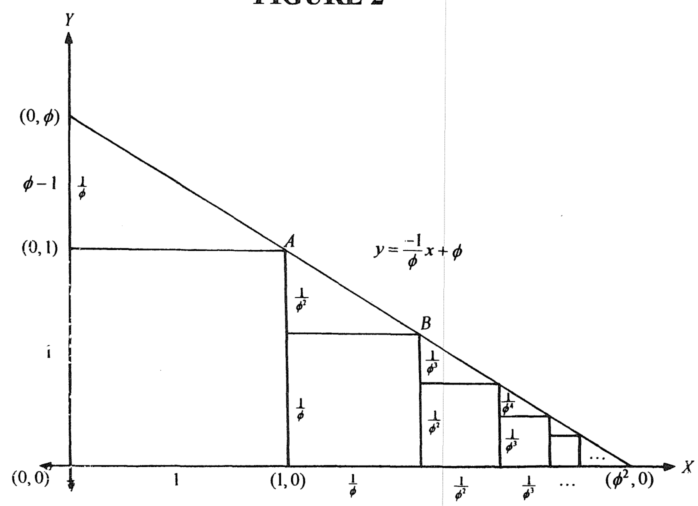

FIGURE 3

The slope of the line through the upper right corners of the n^{th} pair of adjacent squares is

$$\frac{\Delta y}{\Delta x} = \frac{(1/\phi^n - 1/\phi^{n-1})}{(1/\phi^n)} = 1 - \phi = \frac{-1}{\phi}, \quad n = 1, 2, 3, \ldots,$$

making these corners collinear with A and B, the corners of the first pair of adjacent squares. The points $(\phi^2, 0)$ and $(0, \phi)$ satisfy (2) and lie on the golden line, so the sides of the squares on the X-axis provide

$$\sum_{n=0}^{\infty} \frac{1}{\phi^n} = \phi^2. \tag{3}$$

The right triangle under the golden line is a golden triangle and also half of a golden rectangle, since the ratio of its legs is ϕ.

One- and two-dimensional representations of ϕ and its reciprocal can be found in Figure 3. ϕ is the y-intercept of the golden line and, since $\phi^2 = \phi + 1$, the distance between $(1, 0)$ and the x-intercept of the golden line is ϕ. Also ϕ can be seen as the distance from the origin to the end of the second square. $1/\phi$ is the length of the second square and also the altitude of the first triangle. All of the squares fit exactly into the golden rectangle of Figure 2, so the sum of the areas of all of the squares is ϕ, the area of the golden rectangle. The areas of the squares also form a geometric sequence so that

$$\sum_{k=0}^{\infty}\left(\frac{1}{\phi}\right)^{2k} = \phi = \sum_{k=0}^{\infty}(\phi-1)^{2k}. \tag{4}$$

Since the area of the first square is 1, the sum of the areas of the squares beyond the first is $\phi-1$ or $1/\phi$. Thus, we have in one picture both a linear and a planar representation of ϕ and $1/\phi$, neatly sheltered beneath the golden line, $y=(-1/\phi)x+\phi$.

2. EXPANDING THE PICTURE

Above each square, construct a rectangle whose diagonal lies on the golden line shown in Figure 4. Each of these small horizontal rectangles is also a golden rectangle. Joining any such rectangle to the square below it creates a new, larger vertical golden rectangle. Figure 4 also shows the line $y=-x+\phi^2$ drawn through the corners of the golden rectangles.

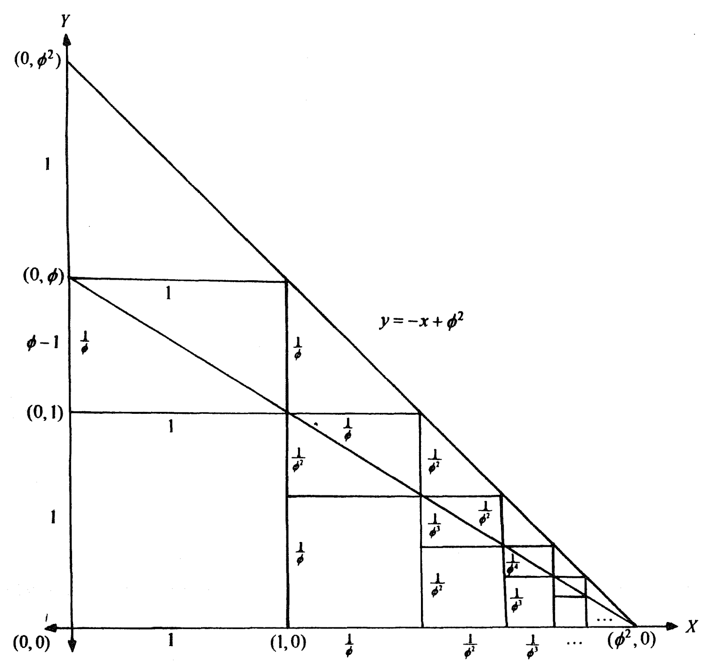

FIGURE 4

Let us focus our attention on the series of vertical and horizontal golden rectangles in Figure 4. After the first, each vertical rectangle is congruent to a horizontal rectangle. The sum ΣV_n of the areas of the vertical rectangles is the area of the largest vertical rectangle added to the sum ΣH_n of the areas of the horizontal rectangles. That is, $\Sigma V_n = \phi + \Sigma H_n$, where also $\Sigma V_n = (\phi^2)\Sigma H_n$ by similarity. Since $\phi^2 = \phi + 1$, $\Sigma V_n = \phi^2$, and $\Sigma H_n = 1$. Also, each vertical rectangle is a square added to a horizontal rectangle. If ΣS_n is the sum of the areas of the squares, $\Sigma V_n = \Sigma S_n + \Sigma H_n$, which gives $\Sigma S_n = \phi$ as in (4). $\Sigma H_n = 1$ leads to

$$\sum_{k=0}^{\infty} \frac{1}{\phi^{2k+1}} = 1, \tag{5}$$

and all of the small rectangles will fit exactly into the first square.

In intercept form, the equation of the line through the upper corners of the horizontal golden rectangle is $(x/\phi^2)+(y/\phi^2)=1$; for the upper corners of the squares, $(x/\phi^2)+(y/\phi)=1$; and for the lower corners of the horizontal rectangles, $(x/\phi^2)+(y/1)=1$.

3. FINDING FIBONACCI

Where we find the golden ratio, we can expect to find Fibonacci, but first we need to set the stage for his entrance. From Figure 4, the length of the side of a square is the same as the length of the adjacent golden rectangle to the right. The length of that vertical golden rectangle is the sum of the length of a square and the width of a horizontal golden rectangle above the square. For example, with the third square and its adjoining golden rectangle, we have $(1/\phi^2)=(1/\phi^3)+(1/\phi^4)$, and for the $(k+1)^{\text{st}}$ square, $(1/\phi)^k=(1/\phi)^{k+1}+(1/\phi)^{k+2}$. This representation of a power of $1/\phi$ as the sum of the next two consecutive powers of $1/\phi$ allows Fibonacci to enter. For consecutive Fibonacci numbers F_j and F_{j+1},

$$\left(\frac{1}{\phi}\right)^k = F_{j+1}\left(\frac{1}{\phi}\right)^{k+j} + F_j\left(\frac{1}{\phi}\right)^{k+j+1}, \tag{6}$$

which can be proved by induction.

Expanding powers of $\phi-1$ and simplifying leads to another expression involving Fibonacci numbers and ϕ, which also can be proved by mathematical induction:

$$(\phi-1)^k = \left(\frac{1}{\phi}\right)^k = \begin{cases} F_k\phi - F_{k+1}, & \text{for } k \text{ odd,} \\ F_{k+1} - F_k\phi, & \text{for } k \text{ even.} \end{cases} \tag{7}$$

To all the beautiful patterns in mathematics, we may now add the golden staircase, the golden line, and all that they inspire.

ACKNOWLEDGMENT

The author would like to thank Dr. Lewis Lum for his encouragement and advice. He would also like to express his appreciation to the anonymous referees who helped to improve the content and readability of this paper.

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AMS Classification Numbers: 11B39, 51M05, 40A05

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