SOME IDENTITIES INVOLVING THE FIBONACCI NUMBERS

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(Submitted January 1996–Final Revision March 1996)

1. INTRODUCTION AND RESULTS

As usual, a second-order linear recurrence sequence $U = (U_n)$, $n = 0, 1, 2, \ldots$, is defined by integers a, b, U_0, U_1 and by the recursion

$$U_{n+2} = bU_{n+1} + aU_n \tag{1}$$

for $n \geq 0$. We suppose that $ab \neq 0$ and not both U_0 and U_1 are zero. If α and β denote the roots of the characteristic polynomial $x^2 - bx - a$ of the sequence U and α / β is not a root of unity, then U is called a nondegenerate sequence. In this case, as is well known (see [2]), the terms of the sequence U can be expressed as $U_n = p\alpha^n - q\beta^n$ for $n = 0, 1, 2, \ldots$, where

$$p = \frac{U_1 - U_0\beta}{\alpha - \beta} \quad \text{and} \quad 1 = \frac{U_1 - U_0\alpha}{\alpha - \beta}.$$

If $U_0 = 0$, $a = b = U_1 = 1$, then the sequence U is called the Fibonacci sequence, and we shall denote it by $F = (F_n)$.

The various properties of second-order linear recurrence sequences were investigated by many authors. For example, Duncan [1] and Kuipers [3] proved that $(\log F_n)$ is uniformly distributed mod 1. Robbing [4] studied the Fibonacci numbers of the forms $px^2 \pm 1$, $px^3 \pm 1$, where p is a prime. The main purpose of this paper is to study how to calculate the summation of one class of second-order linear recurrence sequences, i.e.,

$$\sum_{a_1 + a_2 + \cdots + a_k = n} U_{a_1} U_{a_2} \ldots U_{a_k} \tag{2}$$

where the summation is over all n-tuples with positive integer coordinates $(a_1, a_2, \ldots, a_k)$ such that $a_1 + a_2 + \cdots + a_k = n$.

Regarding (2), it seems that it has not yet been studied; at least this author has not seen expressions like (2) before. The problem is interesting because it can help us to find some new convolution properties. In this paper we use the generating function of the sequence U and its derivative to study the evaluation of (2) and give an interesting identity for any fixed positive integer k. That is, we shall prove the following two propositions.

Proposition 1: Let $U = (U_n)$ be defined by (1). If $U_0 = 0$, then, for any positive integer $k \geq 2$, we have

$$\sum_{a_1 + a_2 + \cdots + a_k = n} U_{a_1} U_{a_2} \ldots U_{a_k} = \frac{U_1^{k-1}}{(b^2 + 4a)^{k-1}(k-1)!}[g_{k-1}(n)U_{n-k+1} + h_{k-1}(n)U_{n-k}],$$

where $g_{k-1}(x)$ and $h_{k-1}(x)$ are two effectively computable polynomials of degree $k - 1$, their coefficients depending only on a, b, and k.

Proposition 2: Under the condition of Proposition 1, we have the following identities:

(i) $\sum_{a+b=n} U_aU_b = \frac{U_1}{b^2+4a}[b(n-1)U_n + 2anU_{n-1}];$

(ii) $\sum_{a+b+c=n} U_aU_bU_c = \frac{U_1^2}{2(b^2+4a)^2}\{[(b^3+4ab)n^2-(3b^3+6ab)n+(2b^3-4ab)]U_{n-1}$
$+[(b^2a+4a^2)n^2-3b^2an+(2b^2a-4a^2)]U_{n-2}\};$

(iii) $\sum_{a+b+c+d=n} U_aU_bU_cU_d = \frac{U_1^3}{6(b^2+4a)^3}\{[(b^5+7b^3a+12ba^2)n^3$
$-(6b^5+30b^3a+24ba^2)n^2+(11b^5+17b^3a-48ba^2)n$
$-(6b^5-30b^3a-36ba^2)]U_{n-2}+[(b^4a+6b^2a^2+8a^3)n^3$
$-(6b^4a+24a^2b^2)n^2+(11b^4a+6b^2a^2-32a^3)n-(6b^4a-36a^2b^2)]U_{n-3}\}.$

Taking $U_1=a=b=1$, then $U_n=F_n$ is the Fibonacci sequence, i.e., $F_0=0$, $F_1=1$, $F_2=1$, $F_3=2$, $F_4=3$, $F_5=5$, $F_6=8$, ... Thus, from Proposition 2, we obtain Corollaries 1 and 2.

Corollary 1: Let (F_n) be the Fibonacci sequence. Then we have:

(i) $\sum_{a+b=n} F_aF_b = \frac{1}{5}[(n-1)F_n+2nF_{n-1}],\ n\geq 1;$

(ii) $\sum_{a+b+c=n} F_aF_bF_c = \frac{1}{50}[(5n^2-9n-2)F_{n-1}+(5n^2-3n-2)F_{n-2}],\ n\geq 2;$

(iii) $\sum_{a+b+c+d=n} F_aF_bF_cF_d = \frac{1}{150}[(4n^3-12n^2-4n+12)F_{n-2}+(3n^3-6n^2-3n+6)F_{n-3}],\ n\geq 3.$

Corollary 2: We have the following congruences:

(i) $(n-1)F_n+2nF_{n-1}\equiv 0 \pmod 5,\ n\geq 1;$

(ii) $(5n^2-9n-2)F_{n-1}+(5n^2-3n-2)F_{n-2}\equiv 0 \pmod{50},\ n\geq 2;$

(iii) $(4n^3-12n^2-4n+12)F_{n-2}+(3n^3-6n^2-3n+6)F_{n-3}\equiv 0 \pmod{150},\ n\geq 3.$

2. PROOF OF THE PROPOSITIONS

In this section, we shall give the proof of the propositions. First, we recall some known results on the second-order linear recurrence sequences and prove two lemmas that will be used in the proof of the propositions.

Let $U=(U_n)$ be a nondegenerate second-order linear recurrence sequence defined by (1). If $U_0=0$, then the generating function of U is

$$G(x)=xF(x)=\frac{U_1x}{1-bx-ax^2}=\sum_{n=0}^{\infty}U_nx^n, \tag{3}$$

where $U_n=G^{(n)}(0)/n!$ and $G^{(k)}(x)$ denotes the k^{th} derivative of $G(x)$.

For $F(x)=G(x)/x=\sum_{n=1}^{\infty}U_nx^{n-1}$, we have the following lemma.

Lemma 1: If $F(x)$ is defined by (3), then $F(x)$ satisfies:

(i) $F^2(x)=\frac{U_1}{b^2+4a}[F'(x)(b+2ax)+4aF(x)];$

(ii) $F^3(x) = \dfrac{U_1^2}{2(b^2+4a)^2}[F''(x)(b+2ax)^2 + 14aF'(x)(b+2ax) + 32a^2F(x)];$

(iii) $F^4(x) = \dfrac{U_1^3}{6(b^2+4a)^3}[F'''(x)(b+2ax)^3 + 30aF''(x)(b+2ax)^2$
$+ 228a^2F'(x)(b+2ax) + 384a^3F(x)].$

Proof: Using the definition of $F(x)$ and the derivative of the function $F(x)(b+2ax)$, we get

$$\begin{aligned}[F(x)(b+2ax)]' &= F'(x)(b+2ax) + 2aF(x) = \left[\frac{U_1(b+2ax)}{1-bx-ax^2}\right]' \\ &= \frac{U_1(b^2+2a+2abx+2a^2x^2)}{(1-bx-ax^2)^2} = \frac{U_1(b^2+4a)}{(1-bx-ax^2)^2} - \frac{2aU_1}{1-bx-ax^2} \\ &= \frac{b^2+4a}{U_1}F^2(x) = 2aF(x),\end{aligned}$$

so that

$$\frac{b^2+4a}{U_1}F^2(x) = F'(x)(b+2ax) + 4aF(x). \tag{4}$$

This gives the conclusion (i) of Lemma 1.

Differentiating in (4), we have

$$\frac{b^2+4a}{U_1}\cdot 2F(x)F'(x) = F''(x)(b+2ax) + 6aF'(x).$$

So

$$\frac{b^2+4a}{U_1}\cdot 2F(x)F'(x)(b+2ax) = F''(x)(b+2ax)^2 + 6aF'(x)(b+2ax).$$

Applying (4) again, we have

$$\frac{b^2+4a}{U_1}\cdot 2F(x)\left[\frac{b^2+4a}{U_1}F^2(x) - 4aF(x)\right] = F''(x)(b+2ax)^2 + 6aF'(x)(b+2ax).$$

Thus,

$$\begin{aligned}\frac{2(b^2+4a)^2}{U_1^2}F^3(x) &= F''(x)(b+2ax)^2 + 6aF'(x)(b+2ax) + \frac{8a(b^2+4a)}{U_1}F^2(x) \\ &= F''(x)(b+2ax)^2 + 14aF'(x)(b+2ax) + 32a^2F(x).\end{aligned} \tag{5}$$

Conclusion (ii) of Lemma 1 now follows from (5).

Similarly, differentiating in (5) and applying (5), we can also obtain

$$\begin{aligned}\frac{3!(b^2+4a)^3}{U_1^3}F^4(x) &= F'''(x)(b+2ax)^3 + 30aF''(x)(b+2ax)^2 \\ &\quad + 228a^2F'(x)(b+2ax) + 384a^3F(x).\end{aligned}$$

This completes the proof of Lemma 1.

Lemma 2: Let $k \geq 2$ be an integer. Then there exist $k-1$ effectively computable positive integers $c_1, c_2, \ldots, c_{k-1}$ such that

$$\frac{(k-1)!(b^2+4a)^{k-1}}{U_1^{k-1}} F^k(x) = F^{(k-1)}(x)(b+2ax)^{k-1} + c_1 a F^{(k-2)}(x)(b+2ax)^{k-2} \\ + \cdots + c_{k-2} a^{k-2} F'(x)(b+2ax) + c_{k-1} a^{k-1} F(x), \tag{6}$$

where $F^{(i)}(x)$ denotes the i^{th} derivative of $F(x)$.

Proof: This formula can be obtained via Lemma 1 and induction.

Now we complete the proof of the propositions. First, we prove Proposition 1. Equating the coefficients of x^{n-k} on both sides of (6), we obtain

$$\frac{(k-1)!(b^2+4a)^{k-1}}{U_1^{k-1}} \sum_{a_1+a_2+\cdots+a_k=n} U_{a_1} U_{a_2} \ldots U_{a_k}$$
$$= \sum_{i=0}^{k-1} c_i a^i \sum_{j=0}^{k-1-i} \binom{k-1-i}{k-1-i-j} \cdot \frac{(n-i-j)!}{(n-k-j)!} b^{k-1-i-j} (2a)^j U_{n-1-i-j}.$$

Substituting $U_{n-m} = bU_{n-m-1} + aU_{n-m-2}$ $(1 \le m \le k-1)$ repeatedly in the above formula gives

$$\frac{(k-1)!(b^2+4a)^{k-1}}{U_1^{k-1}} \sum_{a_1+a_2+\cdots+a_k=n} U_{a_1} U_{a_2} \ldots U_{a_k} = g_{k-1}(n) U_{n-k+1} + h_{k-1}(n) U_{n-k},$$

where $g_{k-1}(x)$ and $h_{k-1}(x)$ are two effectively computable polynomials with their coefficients depending only on a, b and k. This completes the proof of Proposition 1.

To prove Proposition 2, comparing the coefficients of x^{n-2}, x^{n-3}, and x^{n-4} on both sides of Lemma 1, we get the following convolution product formulas:

$$\left(\frac{U_1}{b^2+4a}\right)^{-1} \cdot \left(\sum_{a+b=n} U_a U_b\right) = [b(n-1)U_n + 2anU_{n-1}]; \tag{7}$$

$$\left(\frac{U_1^2}{2(b^2+4a)^2}\right)^{-1} \cdot \left(\sum_{a+b+c=n} U_a U_b U_c\right) \\ = b^2(n^2-3n+2)U_n + ab(4n^2-6n-4)U_{n-1} + 4a^2(n^2-1)U_{n-2}; \tag{8}$$

and

$$\left(\frac{U_1^3}{6(b^2+4a)^3}\right)^{-1} \cdot \left(\sum_{a+b+c+d=n} U_a U_b U_c U_d\right) \\ = b^3(n^3-6n^2+11n-6)U_n + b^2a(6n^3-24n^2+6n+36)U_{n-1} \\ + ba^2(12n^3-24n^2-48n+36)U_{n-2} + a^3(8n^3-32n)U_{n-3}. \tag{9}$$

Substituting $U_n = bU_{n-1} + aU_{n-2}$ in (8), we have

$$\sum_{a+b+c=n} U_a U_b U_c = \frac{U_1^2}{2(b^2+4a)^2} [((b^3+4ab)n^2 - (3b^3+6ab)n + (2b^3-4ab))U_{n-1} \\ + ((b^2a+4a^2)n^2 - 3b^2an + (2b^2a-4a^2))U_{n-2}]. \tag{10}$$

Finally, substituting $U_{n-1} = bU_{n-2} + aU_{n-3}$ and $U_n = bU_{n-1} + aU_{n-2} = (b^2\_a)U_{n-2} + abU_{n-3}$ in (9), we get the identity

$$\begin{aligned}\sum_{a+b+c+d=n} U_aU_bU_cU_d = \frac{U_1^3}{6(b^2+4a)^3}[&((b^5+7b^3a+12ba^2)n^3-(6b^5+30b^3a+24ba^2)n^2\\ &+(11b^5+17b^3a-48ba^2)n-(6b^5-30b^3a-36ba^2))\,U_{n-2}\\ &+((b^4a+6b^2a^2+8a^3)n^3-(6b^4a+24b^2a^2)n^2\\ &+(11b^4a+6b^2a^2-32a^3)n-(6b^4a-36a^2b^2))\,U_{n-3}].\end{aligned} \tag{11}$$

Proposition 2 now follows from (7), (10), and (11).

ACKNOWLEDGMENT

The author expresses his gratitude to the anonymous referee for very helpful and detailed comments that improved the presentation of this paper.

REFERENCES

1. R. I. Duncan. "Application of Uniform Distribution to the Fibonacci Numbers." *The Fibonacci Quarterly* **5.2** (1967):137-40.
2. G. H. Hardy & E. M. Wright. *An Introduction to the Theory of Numbers*. 4th ed. London: Oxford University Press, 1962.
3. L. Kuipers. "Remark on a Paper by R. L. Duncan Concerning the Uniform Distribution Mod 1 of the Sequence of the Logarithms of the Fibonacci Numbers." *The Fibonacci Quarterly* **7.5** (1969):465-66.
4. N. Robbins. "Fibonacci Numbers of the Forms $px^2 \pm 1$ $px^3 \pm 1$, Where p is Prime." In *Applications of Fibonacci Numbers* **1**:77-88. Dordrecht: Kluwer, 1986.

AMS Classification Numbers: 11B37, 11B39

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LETTER TO THE EDITOR

Dear Professor Bergum:

The Fibonacci Quarterly readers will be interested in yet another natural occurrence of the Golden Ratio. This occurrence is described in the current issue of *The College Mathematics Journal* (Vol. 28, No. 3, May 1997). On page 205, Peter Schumer (schumer@middlebury.edu) of Middlebury College in Middlebury VT provides an interesting variant on the classical problem of showing that the rectangle with fixed perimeter and maximum area is a square.

Schumer notes that texts often present this problem as the dilemma of a farmer who has a fixed length of fencing and wants to build the most efficient animal pen for grazing. It is a simple calculus problem. The problem is complicated somewhat when the farmer has a fixed length of fencing and is using one side of a barn for all or part of one side of the animal pen. Schumer provides a neat analysis of the optimum pen shape when the length of fencing is some multiple of the length of the barn side used.

When the length of fencing available is $\sqrt{5}$ times the length of the side of barn used, the optimum pen shape is a golden rectangle. This is a neat result, simply derived, of interest to *FQ* readers, and which I have not seen before.

Best regards,

Harvey J. Hindin

Vice-President, Emerging Technologies Group, Inc.