A SEMIGROUP ASSOCIATED WITH THE k-BONACCI NUMBERS WITH DYNAMIC INTERPRETATION

Víctor F. Sirvent
Departamento de Matemáticas, Universidad Simón Bolívar,
Apartado 89000, Caracas 1086-A, Venezuela
e-mail: vsirvent@usb.ve
(*Submitted February 1996-Final Revision June 1996*)

1. INTRODUCTION

In this paper we shall associate a semigroup with the k-bonacci numbers, which describes the self-similar structure of the dynamical system associated with the substitution $1 \to 12, \dots (k-1) \to 1k, k \to 1$ for $k \geq 3$. The operation that defines the semigroup is used to handle the cylinders of the partitions defined by the self-similar structure of the symbolic system. This collection of cylinders is called the standard partition. The relation between the standard partition and the semigroup is given by Theorem 2.

The dyamical system that arises from this substitution admits geometrical representations as:

- an irrational translation on the $(k-1)$-dimensional torus [7],
- an interval exchange map in the circle [1], and
- a map on a geodesic lamination on the hyperbolic disc [8].

The self-similar structure of the symbolic system is translated to its geometrical realizations. The understanding of the self-similar structure of the symbolic system and its geometric relations on the torus and the circle, using the semigroup, plays an important role in the construction of the geodesic lamination, given in [8], and also in the proofs of other dynamical properties of these systems [8].

2. THE SEMIGROUP

The k-bonacci numbers are obtained by the recurrence relation

$$g_{n+k} = g_{n+k-1} + \cdots + g_{n+1} + g_n \quad \text{for } n \geq 0 \tag{1}$$

with initial conditions $g_j = 2^j$ for $0 \leq j \leq k-1$. We can represent each natural number in a unique way as a sum of the g_i's with no k consecutive g_i's in the present sum. This is a generalization of the Zeckendorf representation of the nonnegative [10] integers using this recurrence relation instead of the Fibonacci relation.

In the rest of this section, we work with the Tribonacci numbers. However, the following constructions and results are valid for all the k-bonacci numbers.

Let n and m be given in the Tribonacci Zeckendorf representation

$$n = \sum_{i=0}^{N} a_i g_i, \quad m = \sum_{j=0}^{M} b_j g_j .$$

Define $n \diamond m$ by

$$n \diamond m = \sum_{i=0}^{N} \sum_{j=0}^{M} a_i b_j g_{i+j}. \tag{2}$$

Unlike the Fibonacci multiplication ([5], [2]), this operation is not associative.

Now we define a new binary operation in $\mathbf{N}$: Let $n = g_{i_0} + \cdots + g_{i_\ell}$, with $g_{i_j} < g_{i_q}$ when $j < q$, be the Zeckendorf representation of n. Observe that we can write n in the following way:

$$\begin{aligned} n &= g_{i_0} \diamond (1 + g_{i_1 - i_0} + \cdots + g_{i_\ell - i_0}) \\ &= g_{i_0} \diamond (1 + g_{i_1 - i_0} \diamond (1 + \cdots + g_{i_\ell - i_1}) \\ &\vdots \\ &= g_{i_0} \diamond (1 + g_{i_1 - i_0} \diamond (1 + \cdots + g_{i_{\ell-1} - i_{\ell-2}} \diamond (1 + g_{i_\ell - i_{\ell-1}}) \cdots)). \end{aligned}$$

Definition 1: Define the binary operation $*$ by

$$\begin{aligned} &\mathbf{N} \times \mathbf{N} \to \mathbf{N} \\ &n * m = g_{i_0} \diamond (1 + g_{i_1 - i_0} \diamond (1 + \cdots + g_{i_{\ell-1} - i_{\ell-2}} \diamond (1 + g_{i_\ell - i_{\ell-1}} \diamond m) \cdots)). \end{aligned}$$

Properties:

- $1 * m = m * 1 = m$.
- If $n = g_q$ then $n * m = g_q \diamond m$.
- $*$ is not commutative: e.g., $9 * 3 = 22$ and $3 * 9 = 18$.
- In general, it is not associative: e.g., $3 * (3 * 2) = 10$ and $(3 * 3) * 2 = 13$.

For this reason, we keep the following convention:

$$m_1 * m_2 * \cdots * m_\ell \stackrel{\text{def}}{=} m_1 * (m_2 * (\cdots * (m_{\ell-2} * (m_{\ell-1} * m_\ell)) \cdots)).$$

However, this operation is associative in a subset of the natural numbers. Let $n_1 = g_1 = 2$, $n_2 = g_0 + g_2 = 1 + 4 = 5$, $n_3 = g_0 + g_1 + g_3 = 1 + 2 + 7 = 10$, $n_0 = g_0 = 1$, and $\mathcal{P}$ the set generated by n_1, n_2, n_3 under the operation $*$, i.e.,

$$\mathcal{P}_\ell = \{n_{i_1} * \cdots * n_{i_\ell} | i_j = 1, 2, \text{ or } 3 \text{ for all } j\}, \quad \mathcal{P}_0 = \{1\}, \quad \mathcal{P} = \cup_{\ell \geq 0} \mathcal{P}_\ell .$$

Given any three natural numbers n, m, and m' then the associativity in $n * m * m'$ fails when we do the operation $n * m$ and we get an expression with three consecutive g_i's and, therefore, we have to use the relation (1) to express the number according to the Zeckendorf representation.

Easy calculations show that when we compute $n_i * n_j$ for $i, j = 0, 1, 2, 3$ we never get three consecutive g_i's. So the operation $* : \mathcal{P} \times \mathcal{P} \to \mathcal{P}$ is associative. Therefore, we have proved

Theorem 1: $(\mathcal{P}, *)$ is a semigroup.

3. THE SUBSTITUTION

A substitution in a finite alphabet $\mathcal{A}$ is a map, Π, from the alphabet to a set of words in this alphabet. This map extends to a map from the set of words in the alphabet $\mathcal{A}$ into itself by juxtaposition, i.e., $\Pi(UV) = \Pi(U)\Pi(V)$, where U and V are words in the alphabet and $\Pi(\emptyset) = \emptyset$. In this way, the substitution is extended to the set of infinite sequences in the alphabet $\mathcal{A}$. See [6] for an introduction to the theory of substitutions. In this paper we consider the substitution

$$\Pi : \{1, 2, \ldots, n\}^{\mathbf{N}^*} \to \{1, 2, \ldots, n\}^{\mathbf{N}^*}$$

$$1 \xrightarrow{\Pi} 12, 2 \xrightarrow{\Pi} 13, \ldots, (k-1) \xrightarrow{\Pi} 1k, k \xrightarrow{\Pi} 1. \tag{3}$$

This substitution is Pisot, since the Perron-Frobenius eigenvalue of the matrix that represents the substitution is a Pisot number. A *Pisot number* is an algebraic integer such that all its Galois conjugates are strictly inside the unit circle [3].

The map Π has a unique fixed point $\underline{\mathbf{u}} = u_0 u_1 \ldots$. We consider the closure, in the product topology on $\{1, 2, \ldots, k\}^{\mathbf{N}^*}$, of the orbit under the shift map—$\sigma(u_0u_1u_2\ldots) = u_1u_2\ldots$—of the fixed point. This space is denoted by Ω. The dynamical system (Ω, σ) is minimal. The dynamical and geometrical properties of this substitution have been studied in [7], [1], [4], [8], [9].

Note that the relation of this substitution to the k-bonacci numbers is the following: if $|V|$ denotes the length of the word V, and $g_j = |\Pi^j(1)|$, we have the recurrence relation $g_{n+k} = g_{n+k-1} + \cdots + g_n$, since the substitution satisfies

$$\Pi^{n+k}(1) = \Pi^{n+k-1}(1)\Pi^{n+k-2}(1)\cdots\Pi^{n+1}(1)\Pi^n(1)\ \forall n \geq 0.$$

The space Ω admits a natural self-similar partition $\{\Omega_1, \ldots, \Omega_k\}$, where $\Omega_i = \{\underline{v} \in \Omega | v_0 = i\}$. The self-similarity among the elements of this partition comes from the commutativity of the diagram:

$$\begin{array}{ccc} \Omega & \xrightarrow{\sigma} & \Omega \\ {\scriptstyle \Pi}\downarrow & & \downarrow{\scriptstyle \Pi} \\ \Omega_1 & \xrightarrow{\tilde{\sigma}} & \Omega_1 \end{array}$$

where $\tilde{\sigma}$ denotes the induced map of σ on Ω_1, i.e.,

$$\tilde{\sigma}(\underline{v}) = \sigma^{\min\{\ell | \sigma^\ell(\underline{v}) \in \Omega_1\}}(\underline{v}).$$

In the rest of the paper we will assume that $k = 3$. However, the results are valid for $k \geq 3$.

We are going to show how to express $\sigma^n(\underline{\mathbf{u}})$ as a composition of powers of Π, applied to $\sigma(\underline{\mathbf{u}})$, and σ (without using its powers). In particular, we shall associate with each natural number n an operator $O_{\sigma,\Pi}(n)$ such that $\sigma^n(\underline{\mathbf{u}}) = O_{\sigma,\Pi}(n)(\sigma(\underline{\mathbf{u}}))$. Moreover, we shall prove the property

$$O_{\sigma,\Pi}(m) \circ O_{\sigma,\Pi}(n) = O_{\sigma,\Pi}(m * n).$$

Definition 2: Let $n = g_{i_0} + \cdots + g_{i_\ell}$ be the representation of n according to the recurrence relation (1). Then

$$n = g_{i_0} \diamond (1 + g_{i_1 - i_0} \diamond (1 + \cdots + g_{i_{\ell-1} - i_{\ell-2}} \diamond (1 + g_{i_\ell - i_{\ell-1}}) \cdots)).$$

We define

$$O_{\sigma,\Pi}(n) : \Omega \to \Omega;$$

$$O_{\sigma,\Pi}(n) = \Pi^{i_0}\sigma\Pi^{i_1 - i_0}\sigma \cdots \Pi^{i_{\ell-1} - i_{\ell-2}}\sigma\Pi^{i_\ell - i_{\ell-1}}.$$

Lemma 1: The map $O_{\sigma,\Pi}(n)$ satisfies the properties:

(a) $O_{\sigma,\Pi}(n)(\sigma(\underline{\mathbf{u}})) = \sigma^n(\underline{\mathbf{u}})$ for any $n \in \mathbf{N}$.

(b) $O_{\sigma,\Pi}(m) \circ O_{\sigma,\Pi}(n) = O_{\sigma,\Pi}(m * n)$ for $m, n \in \mathcal{P}$.

First, we are going to prove the following proposition.

Proposition 1: Let g_q be the q^{th} Tribonacci number, then

(a) $\sigma^{g_q}(\underline{\mathbf{u}}) = \Pi^q(\sigma(\underline{\mathbf{u}}))$,

(b) $\widetilde{\sigma}^{g_q}(\underline{\mathbf{u}}) = \sigma^{g_{q+1}}(\underline{\mathbf{u}})$,

(c) $\sigma^{g_q \diamond n}(\underline{\mathbf{u}}) = \sigma^{n \diamond g_q}(\underline{\mathbf{u}}) = \Pi^q \sigma^n(\underline{\mathbf{u}})$ for all $n \in \mathbf{N}^*$.

Proof of Proposition 1:

(a) This fact is proved by induction on q. In the case $q = 1$:

$$\underline{\mathbf{u}} = u_0 \sigma(\underline{\mathbf{u}}) = 1\sigma(\underline{\mathbf{u}}) \text{ so } \underline{\mathbf{u}} = \Pi(\underline{\mathbf{u}}) = \Pi(1)\Pi(\sigma(\underline{\mathbf{u}})) = 12\Pi(\sigma(\underline{\mathbf{u}})).$$

Therefore, $\sigma^2(\underline{\mathbf{u}}) = \Pi\sigma(\underline{\mathbf{u}})$ but $2 = g_1$. Hence, $\sigma^{g_1}(\underline{\mathbf{u}}) = \Pi(\sigma(\underline{\mathbf{u}}))$. Let the expression of $\underline{\mathbf{u}}$ be given as

$$\underline{\mathbf{u}} = u_0 \dots u_{g_q - 1}\sigma^{g_q}(\underline{\mathbf{u}}) = \Pi^q(1)\sigma^{g_q}(\underline{\mathbf{u}}).$$

Since $\underline{\mathbf{u}}$ is the fixed point of the substitution, we have $\underline{\mathbf{u}} = \Pi^{q+1}(1)\Pi(\sigma^{g_q}(\underline{\mathbf{u}}))$. Therefore, we have

$$\sigma^{g_{q+1}}(\underline{\mathbf{u}}) = \Pi(\sigma^{g_q}(\underline{\mathbf{u}})) = \Pi(\Pi^q(\sigma(\underline{\mathbf{u}})) = \Pi^{q+1}(\sigma(\underline{\mathbf{u}})).$$

(b) As we showed in part (a) of this proposition, $\sigma^{g_{q+1}}(\underline{\mathbf{u}}) = \Pi(\sigma^{g_q}(\underline{\mathbf{u}}))$. Since $\Pi \circ \sigma = \widetilde{\sigma} \circ \Pi$, we have

$$\Pi(\sigma^{g_q}(\underline{\mathbf{u}})) = \widetilde{\sigma}^{g_q}(\Pi(\underline{\mathbf{u}}))$$

and, since $\underline{\mathbf{u}}$ is the fixed point of the substitution, we have $\sigma^{g_{q+1}}(\underline{\mathbf{u}}) = \widetilde{\sigma}^{g_q}(\underline{\mathbf{u}})$.

(c) Let $n = g_{i_0} + \cdots + g_{i_\ell}$. We can write $\underline{\mathbf{u}} = u_0 \dots u_{n-1}\sigma^n(\underline{\mathbf{u}})$; according to [7]:

$$u_0 \dots u_{n-1} = \Pi^{i_\ell}(1) \cdots \Pi^{i_0}(1),$$

and using the fact that $\underline{\mathbf{u}}$ is a fixed point of the substitution Π, we have

$$\underline{\mathbf{u}} = \Pi^q(\underline{\mathbf{u}}) = \Pi^{i_\ell + q}(1) \cdots \Pi^{i_0 + q}(1)\Pi^q\sigma^n(\underline{\mathbf{u}}).$$

Therefore,

$$\Pi^q\sigma^n(\underline{\mathbf{u}}) = \sigma^{g_{i_\ell + q} + \cdots + g_{i_0 + q}}(\underline{\mathbf{u}}) = \sigma^{g_q \diamond n}(\underline{\mathbf{u}}). \text{ Q.E.D.}$$

Proof of Lemma 1:

(a) Let

$$\begin{aligned} n &= g_{i_0} + \cdots + g_{i_\ell} \\ &= g_{i_0} \diamond (1 + g_{i_1 - i_0} \diamond (1 + \cdots + g_{i_{\ell-1} - i_{\ell-2}} \diamond (1 + g_{i_\ell - i_{\ell-1}}) \cdots)). \end{aligned}$$

By Proposition 1,

$$\begin{aligned} \Pi^{i_\ell - i_{\ell-1}}(\sigma(\underline{\mathbf{u}})) &= \sigma^{g_{i_\ell - i_{\ell-1}}}(\underline{\mathbf{u}}) \\ \sigma\Pi^{i_\ell - i_{\ell-1}}(\sigma(\underline{\mathbf{u}})) &= \sigma^{1 + g_{i_\ell - i_{\ell-1}}}(\underline{\mathbf{u}}) \\ \Pi^{i_{\ell-1} - i_{\ell-2}}\sigma\Pi^{i_\ell - i_{\ell-1}}(\sigma(\underline{\mathbf{u}})) &= \sigma^{g_{i_{\ell-1} - i_{\ell-2}} \diamond (1 + g_{i_\ell - i_{\ell-1}})}(\underline{\mathbf{u}}) \\ &\vdots \\ \Pi^{i_0}\sigma\Pi^{i_1 - i_0}\sigma \cdots \Pi^{i_\ell - i_{\ell-1}}(\sigma(\underline{\mathbf{u}})) &= \sigma^{g_{i_0} \diamond (1 + g_{i_1 - i_0} \diamond (1 + \cdots + g_{i_{\ell-1} - i_{\ell-2}} \diamond (1 + g_{i_\ell - i_{\ell-1}}) \cdots))}(\underline{\mathbf{u}}). \end{aligned}$$

But the last term is $\sigma^n(\underline{\mathbf{u}})$ by using the expression for n given at the beginning of the proof. But, by Definition 2, $O_{\sigma,\Pi}(n)=\Pi^{i_0}\sigma\Pi^{i_1-i_0}\sigma\cdots\Pi^{i_\ell-i_{\ell-1}}$. Hence, $O_{\sigma,\Pi}(n)(\sigma(\underline{\mathbf{u}}))=\sigma^n(\underline{\mathbf{u}})$.

(b) Let

$$\begin{aligned} m &= g_{j_0}+\cdots+g_{j_q} \quad \text{and } m\in\mathcal{P} \\ &= g_{j_0}\diamond(1+g_{j_1-j_0}\diamond(1+\cdots+g_{j_{q-1}-j_{q-2}}\diamond(1+g_{j_q-j_{q-1}})\cdots)). \end{aligned}$$

So

$$O_{\sigma,\Pi}(m)=\Pi^{j_0}\sigma\Pi^{j_1-j_0}\sigma\cdots\Pi^{j_q-j_{q-1}}$$

and

$$O_{\sigma,\Pi}(m)\circ O_{\sigma,\Pi}(n)=$$
$$\underbrace{\Pi^{j_0}\sigma\Pi^{j_1-j_0}\sigma\cdots\Pi^{j_q-j_{q-1}}}_{O_{\sigma,\Pi}(m)}\underbrace{\Pi^{i_0}\sigma\Pi^{i_1-i_0}\sigma\cdots\Pi^{i_\ell-i_{\ell-1}}}_{O_{\sigma,\Pi}(n)}.$$

Since m and $n\in\mathcal{P}$,

$$m*n=g_{j_0}\diamond(1+\cdots\diamond(1+g_{j_q-j_{q-1}}\diamond g_{i_0}\diamond(1+g_{i_1-i_0}\diamond(1+\cdots\diamond(1+g_{i_\ell-i_{\ell-1}})\cdots)))).$$

Therefore,

$$O_{\sigma,\Pi}(m*n)=O_{\sigma,\Pi}(m)\circ O_{\sigma,\Pi}(n).\quad \text{Q.E.D.}$$

Each of the sets Ω_i is similar to Ω:

$$\begin{aligned} \Omega_1 &= \Pi(\Omega); \\ \Omega_2 &= \sigma(\Pi^2(\Omega))=\sigma(\Pi(\Omega_1)); \\ \Omega_3 &= \sigma(\Pi(\sigma(\Pi^2(\Omega))))=\sigma(\Pi(\sigma(\Pi(\Omega_1))))=\sigma(\Pi(\Omega_2)). \end{aligned}$$

This similarity induces a partition in each of the Ω_i's and each of these cylinders can be subdivided into three subcylinders according to the maps Π, $\sigma\Pi^2$, and $\sigma\Pi\sigma\Pi^2$.

Definition 3: The collection of subsets of Ω generated by the system of iterated maps (Π, $\sigma\Pi^2$, $\sigma\Pi\sigma\Pi^2$) is called the *standard partition of* Ω. The elements of this collection are called *cylinders.*

Theorem 2: R is a cylinder of the standard partition if and only if there exists an element n of $\mathcal{P}$ such that $R=O_{\sigma,\Pi}(n)(\Omega)$.

Proof: Let n be an element of $\mathcal{P}$ so $n=n_{i_0}*\cdots*n_{i_\ell}$, where $i_j\in\{1,2,3\}$, then $O_{\sigma,\Pi}(n)=O_{\sigma,\Pi}(n_{i_0})\cdots O_{\sigma,\Pi}(n_{i_\ell})$, since $n_1=g_1$, $n_2=g_0+g_2$, $n_3=g_0+g_1+g_3=g_0+g_1\diamond(g_0+g_2)$, and we have

$$O_{\sigma,\Pi}(n_1)=\Pi,\quad O_{\sigma,\Pi}(n_2)=\sigma\Pi^2,\quad O_{\sigma,\Pi}(n_3)=\sigma\Pi\sigma\Pi^2.$$

Hence, $O_{\sigma,\Pi}(n)(\Omega)$ is a cylinder of the standard partition.

Reciprocally, given R a cylinder of the standard partition, by construction it is equal to a composition of $\Pi, \sigma\Pi^2$, and $\sigma\Pi\sigma\Pi^2$, i.e., it is of the form $O_{\sigma,\Pi}(n_{i_0})O_{\sigma,\Pi}(n_{i_1})\cdots O_{\sigma,\Pi}(n_{i_k})$. By Lemma 1, we have that $R = O_{\sigma,\Pi}(m)(\Omega)$, where $m = n_{i_0} * \cdots * n_{i_k}$. Q.E.D.

ACKNOWLEDGMENTS

The author would like to thank Anthony Manning, who was the supervisor of the author's Ph.D. thesis. The results of this paper are contained in this thesis.

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AMS Classification Numbers: 11B39, 11Z05