THE $(2,T)$ GENERALIZED FIBONACCI SEQUENCES

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Recently the $(2, F)$ and $(3, F)$ generalized Fibonacci sequences were considered and the generating functions for these sequences were derived (see [1] through [8]). The purpose of this note is to derive generating functions for the $(2, T)$ generalized Tribonacci sequences.

Let $S = (a, b)$ and S_b be the group of permutations on S. Let i be the identity and $\alpha = (a, b)$. Let τ_i be a permutation of S_b for $0 \le i \le 2$ and $Y_i = \{a_i, b_i\}$ for $i \ge 0$. Finally, let a_1, a_2, a_3, b_1, b_2, and b_3 be six distinct real numbers. Then

$$Y_{n+3} = \sum_{i=0}^{2} \tau_i Y_{n+i}, \quad n \ge 0, \tag{1}$$

with initial conditions $Y_i = \{a_i, b_i\}$ for $0 \le i \le 2$, are the eight systems of third-order difference equations defining the $(2, T)$ generalized Tribonacci sequences.

Define

$$\delta_i = \begin{cases} 0 & \text{if } \tau_i = i, \\ 1 & \text{if } \tau_i = (a, b), \end{cases}$$

and $S = \sum_{i=0}^{2} \delta_i 2^i$. Then each of the eight systems (1) corresponds to an integer S where $0 \le s \le 7$. When S is expressed as a binary number and the right-hand member of (1) is arranged in descending order of subscripts, then the 1's in the binary number indicate the position(s) of the elements b_i in the equation for a_n and the position(s) of the elements a_i in the equation for b_n. If $s = 0 = 000_2$ the system is

$$a_{n+3} = a_{n+2} + a_{n+1} + a_n,$$
$$b_{n+3} = b_{n+2} + b_{n+1} + b_n.$$

In this case the $(2, T)$ generalized Fibonacci sequences are a pair of generalized Tribonacci sequences. This case is excluded from further consideration.

Consider the seven difference systems

$$iY^s_{n+3} = \tau_2 Y^s_{n+2} + \tau_1 Y^s_{n+1} + \tau_0 Y^s_n, \quad 1 \le s \le 7, \tag{2}$$

with initial conditions

$$Y^s_i = \{a_i, b_i\}, \quad 0 \le i \le 2.$$

Atanassov [3] proved that these systems are equivalent to seven sixth-order systems

$$\sum_{i=0}^{6} k^s_i a^s_{n+6-i} = 0, \quad \sum_{i=0}^{6} k^s_i b^s_{n+6-i} = 0, \quad n \ge 0, \tag{3}$$

with initial conditions $\langle a_i \rangle_0^5$ and $\langle b_i \rangle_0^5$, respectively. The values for k^s_i for $1 \le s \le 7$ and $0 \le i \le 6$ are given in Table 1.

TABLE 1. Values of k_i^s

s	0	1	2	3	4	5	6
1	1	−2	−1	2	1	0	−1
2	1	−2	1	−2	1	0	1
3	1	−2	1	0	−1	−2	−1
4	1	0	−3	−2	1	2	1
5	1	0	−3	0	−1	0	−1
6	1	0	−1	−4	−1	0	1
7	1	0	−1	−2	−3	−2	−1

Let $p^s(x)=\sum_{i=0}^{6}k_i^s x^i$ and let $\{P_j^s\}_{j=0}^{\infty}$ be the recursive numbers (of order six) determined by $1/p^s(x)$. Then the seven recursion relations and first terms of the sequences are given in Table 2.

TABLE 2

S	Recursive Relations	First 7 Terms
1	$P_{n+6}=2P_{n+5}+P_{n+4}-2P_{n+3}-P_{n+2}+P_n$	1 2 5 10 20 38 72
2	$P_{n+6}=2P_{n+5}-P_{n+4}+2P_{n+3}-P_{n+2}-P_n$	1 2 3 6 12 22 40
3	$P_{n+6}=2P_{n+5}+P_{n+4}+P_{n+2}+2P_{n+1}+P_n$	1 2 3 4 6 12 26
4	$P_{n+6}=3P_{n+4}+2P_{n+3}-P_{n+2}-2P_{n+1}-P_n$	1 0 3 2 8 10 24
5	$P_{n+6}=3P_{n+4}+P_{n+2}+P_n$	1 0 3 0 10 0 34
6	$P_{n+6}=P_{n+4}+4P_{n+3}+P_{n+2}-P_n$	1 0 1 4 2 8 18
7	$P_{n+6}=P_{n+4}+2P_{n+3}+3P_{n+2}+2P_{n+1}+P_n$	1 0 1 2 4 6 12

Let $f^s(x)$ and $g^s(x)$ be the solutions to the seven systems and let

$$f^s(x)=\sum_{j=0}^{\infty}a_j^s x^j \quad \text{and} \quad g^s(x)=\sum_{j=0}^{\infty}b_j^s x^j.$$

Substituting $f^s(x)$ into the difference systems (3) yields

$$f^s(x)=\left(\sum_{i=0}^{5}q_i^s x^i\right)\left(\sum_{j=0}^{\infty}P_j^s x^j\right),$$

where P_j^s are from the sequences in Table 2 and $q_i^s=\sum_{m=0}^{i}k_m^s a_{i-m}^s$, $0\le i\le 5$.

Expanding and collecting terms gives

$$f^s(x)=\sum_{j=0}^{4}\left(\sum_{i=0}^{j}q_i^s P_{j-i}^s\right)x^j+\sum_{j=5}^{\infty}\sum_{i=0}^{5}(q_i^s P_{j-i}^s)\,x^j$$

for the generating function of $\{a_i^s\}_0^{\infty}$. The terms of the sequence are given by

$$a_j^s=\sum_{i=0}^{j}q_i^s P_{j-i}^s=\sum_{i=0}^{j}\left[\sum_{m=0}^{i}k_m^s a_{i-m}^s\right]P_{j-i}^s \quad \text{for } j<5,$$

and

$$a_j^s = \sum_{i=0}^{5} q_i^s P_{j-i}^s = \sum_{i=0}^{5} \left[\sum_{m=0}^{i} k_m^s a_{i-m}^s \right] P_{j-i}^s, \quad \text{for } j \geq 5.$$

The values of a_i^s, $3 \leq i \leq 5$, are computed in terms of a_0^s, a_1^s, a_2^s, b_0^s, b_1^s, and b_2^s by use of equations (2). The sequences $\{b_i^s\}_0^\infty$ have the same form for each s.

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