

THE $(2, T)$ GENERALIZED FIBONACCI SEQUENCES

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Recently the $(2, F)$ and $(3, F)$ generalized Fibonacci sequences were considered and the generating functions for these sequences were derived (see [1] through [8]). The purpose of this note is to derive generating functions for the $(2, T)$ generalized Tribonacci sequences.

Let $S = (a, b)$ and S_b be the group of permutations on S . Let i be the identity and $\alpha = (a, b)$. Let τ_i be a permutation of S_b for $0 \leq i \leq 2$ and $Y_i = \{a_i, b_i\}$ for $i \geq 0$. Finally, let $a_1, a_2, a_3, b_1, b_2,$ and b_3 be six distinct real numbers. Then

$$Y_{n+3} = \sum_{i=0}^2 \tau_i Y_{n+i}, \quad n \geq 0, \quad (1)$$

with initial conditions $Y_i = \{a_i, b_i\}$ for $0 \leq i \leq 2$, are the eight systems of third-order difference equations defining the $(2, T)$ generalized Tribonacci sequences.

Define

$$\delta_i = \begin{cases} 0 & \text{if } \tau_i = i, \\ 1 & \text{if } \tau_i = (a, b), \end{cases}$$

and $S = \sum_{i=0}^2 \delta_i 2^i$. Then each of the eight systems (1) corresponds to an integer S where $0 \leq S \leq 7$. When S is expressed as a binary number and the right-hand member of (1) is arranged in descending order of subscripts, then the 1's in the binary number indicate the position(s) of the elements b_i in the equation for a_n and the position(s) of the elements a_i in the equation for b_n . If $s = 0 = 000_2$ the system is

$$\begin{aligned} a_{n+3} &= a_{n+2} + a_{n+1} + a_n, \\ b_{n+3} &= b_{n+2} + b_{n+1} + b_n. \end{aligned}$$

In this case the $(2, T)$ generalized Fibonacci sequences are a pair of generalized Tribonacci sequences. This case is excluded from further consideration.

Consider the seven difference systems

$$iY_{n+3}^s = \tau_2 Y_{n+2}^s + \tau_1 Y_{n+1}^s + \tau_0 Y_n^s, \quad 1 \leq s \leq 7, \quad (2)$$

with initial conditions

$$Y_i^s = \{a_i, b_i\}, \quad 0 \leq i \leq 2.$$

Atanassov [3] proved that these systems are equivalent to seven sixth-order systems

$$\sum_{i=0}^6 k_i^s a_{n+6-i}^s = 0, \quad \sum_{i=0}^6 k_i^s b_{n+6-i}^s = 0, \quad n \geq 0, \quad (3)$$

with initial conditions $\langle a_i \rangle_0^s$ and $\langle b_i \rangle_0^s$, respectively. The values for k_i^s for $1 \leq s \leq 7$ and $0 \leq i \leq 6$ are given in Table 1.

TABLE 1. Values of k_i^s

s	0	1	2	3	4	5	6
1	1	-2	-1	2	1	0	-1
2	1	-2	1	-2	1	0	1
3	1	-2	1	0	-1	-2	-1
4	1	0	-3	-2	1	2	1
5	1	0	-3	0	-1	0	-1
6	1	0	-1	-4	-1	0	1
7	1	0	-1	-2	-3	-2	-1

Let $p^s(x) = \sum_{i=0}^6 k_i^s x^i$ and let $\{P_j^s\}_{j=0}^\infty$ be the recursive numbers (of order six) determined by $1/p^s(x)$. Then the seven recursion relations and first terms of the sequences are given in Table 2.

TABLE 2

S	Recursive Relations	First 7 Terms
1	$P_{n+6} = 2P_{n+5} + P_{n+4} - 2P_{n+3} - P_{n+2} + P_n$	1 2 5 10 20 38 72
2	$P_{n+6} = 2P_{n+5} - P_{n+4} + 2P_{n+3} - P_{n+2} - P_n$	1 2 3 6 12 22 40
3	$P_{n+6} = 2P_{n+5} + P_{n+4} + P_{n+2} + 2P_{n+1} + P_n$	1 2 3 4 6 12 26
4	$P_{n+6} = 3P_{n+4} + 2P_{n+3} - P_{n+2} - 2P_{n+1} - P_n$	1 0 3 2 8 10 24
5	$P_{n+6} = 3P_{n+4} + P_{n+2} + P_n$	1 0 3 0 10 0 34
6	$P_{n+6} = P_{n+4} + 4P_{n+3} + P_{n+2} - P_n$	1 0 1 4 2 8 18
7	$P_{n+6} = P_{n+4} + 2P_{n+3} + 3P_{n+2} + 2P_{n+1} + P_n$	1 0 1 2 4 6 12

Let $f^s(x)$ and $g^s(x)$ be the solutions to the seven systems and let

$$f^s(x) = \sum_{j=0}^\infty a_j^s x^j \quad \text{and} \quad g^s(x) = \sum_{j=0}^\infty b_j^s x^j.$$

Substituting $f^s(x)$ into the difference systems (3) yields

$$f^s(x) = \left(\sum_{i=0}^5 q_i^s x^i \right) \left(\sum_{j=0}^\infty P_j^s x^j \right),$$

where P_j^s are from the sequences in Table 2 and $q_i^s = \sum_{m=0}^i k_m^s a_{i-m}^s$, $0 \leq i \leq 5$.

Expanding and collecting terms gives

$$f^s(x) = \sum_{j=0}^4 \left(\sum_{i=0}^j q_i^s P_{j-i}^s \right) x^j + \sum_{j=5}^\infty \sum_{i=0}^5 (q_i^s P_{j-i}^s) x^j$$

for the generating function of $\{a_i^s\}_0^\infty$. The terms of the sequence are given by

$$a_j^s = \sum_{i=0}^j q_i^s P_{j-i}^s = \sum_{i=0}^j \left[\sum_{m=0}^i k_m^s a_{i-m}^s \right] P_{j-i}^s \quad \text{for } j < 5,$$

and

$$\alpha_j^s = \sum_{i=0}^5 q_i^s P_{j-i}^s = \sum_{i=0}^5 \left[\sum_{m=0}^i k_m^s \alpha_{i-m}^s \right] P_{j-i}^s, \quad \text{for } j \geq 5.$$

The values of α_i^s , $3 \leq i \leq 5$, are computed in terms of α_0^s , α_1^s , α_2^s , b_0^s , b_1^s , and b_2^s by use of equations (2). The sequences $\{b_i^s\}_0^\infty$ have the same form for each s .

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