COMBINATORIAL EXPRESSIONS FOR LUCAS NUMBERS

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1. AIM OF THIS NOTE

Several closed-form expressions involving binomial coefficients exist for Lucas numbers. The most celebrated among them is the following specialization of *Waring's formula* (e.g., see (6) of [7]),

$$L_n = \sum_{i=0}^{\lfloor n/2 \rfloor} \frac{n}{n-i}\binom{n-i}{i} \quad (n \geq 1), \tag{1.1}$$

where the symbol $\lfloor \cdot \rfloor$ denotes the greatest integer function. Other combinatorial expressions for Lucas numbers are:

$$L_n = \frac{1}{2^{n-1}} \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n}{2i} 5^i \quad (n \geq 0) \quad \text{(e.g., see (4) of [7]);} \tag{1.2}$$

$$L_n = \sum_{i=-\lfloor (n+1)/5 \rfloor}^{\lfloor n/5 \rfloor} (-1)^i \frac{n + \lfloor (n-5i)/2 \rfloor}{n} \binom{n}{\lfloor (n-5i)/2 \rfloor} \quad (n \geq 1); \tag{1.3}$$

$$L_{2n} = \sum_{i=0}^{n} (-1)^i \frac{2n}{2n-i} \binom{2n-i}{i} 5^{n-i} \quad (n \geq 1) \quad \text{(from (4.2) of [2]).} \tag{1.4}$$

A supposedly new combinatorial expression for odd-subscripted Lucas numbers is reported (without proof) in the Appendix.

Expression (1.3) was obtained by Robbins [7] on the basis of an analogous formula for Fibonacci numbers that was established by Andrews in [1].

As reported in (1.5) of [4], Jaiswal [3] discovered that

$$F_{n+3} = 1 + \sum_{i=0}^{\lfloor n/3 \rfloor} (-1)^i \binom{n-2i}{i} 2^{n-3i} \quad (n \geq 0). \tag{1.5}$$

The aim of this note is to parallel Robbins' work by using (1.5) to prove a new combinatorial expression for Lucas numbers that can be added to the above list.

2. ANOTHER COMBINATORIAL EXPRESSION FOR L_n

We discovered that

$$L_n = -1 + \sum_{i=0}^{\lfloor n/3 \rfloor} (-1)^i \frac{n}{n-2i} \binom{n-2i}{i} 2^{n-3i} \quad (n \geq 1). \tag{2.1}$$

Proof: Let us write

$$\begin{aligned}
S_n &\overset{\text{def}}{=} \sum_{i=0}^{\lfloor n/3 \rfloor} (-1)^i \frac{n}{n-2i}\binom{n-2i}{i} 2^{n-3i} \\
&= \sum_{i=0}^{\lfloor n/3 \rfloor} (-1)^i \left[\binom{n-2i}{i} + 2\binom{n-1-2i}{i-1}\right] 2^{n-3i} \\
&= F_{n+3} - 1 + 2\sum_{i=0}^{\lfloor n/3 \rfloor} (-1)^i \binom{n-1-2i}{i-1} 2^{n-3i} \qquad \text{[from (1.5)]} \\
&= F_{n+3} - 1 + 2\sum_{j=-1}^{\lfloor n/3 \rfloor -1} (-1)^{j+1} \binom{n-3-2j}{j} 2^{n-3-3j}.
\end{aligned} \tag{2.2}$$

Since $\lfloor n/3 \rfloor - 1 = \lfloor (n-3)/3 \rfloor$ and the binomial coefficient in (2.2) vanishes for $j = -1$ (see [6], p. 2), (2.2) can be rewritten as

$$\begin{aligned}
S_n &= F_{n+3} - 1 - 2\sum_{j=0}^{\lfloor (n-3)/3 \rfloor} (-1)^j \binom{n-3-2j}{j} 2^{n-3-3j} \\
&= F_{n+3} - 2F_n + 1 = L_n + 1 \quad \text{[from (1.5)]}.
\end{aligned}$$

3. CONCLUDING REMARKS

Some simple divisibility and congruence properties of the Lucas numbers can be derived immediately from their closed-form expressions. For example, from (1.1), it can be seen that $L_p \equiv 1 \pmod{p}$ (p a prime), whereas, from (1.2), it is apparent that no Lucas number is divisible by 5.

From (2.1), it is evident that L_n is even iff $n \equiv 0 \pmod{3}$. More precisely, it is not hard to see that

$$L_n \equiv 3^{1-r} x_r (-1)^{\lfloor n/3 \rfloor} - 1 \pmod{2^{r+3}}, \tag{3.1}$$

where r is the residue of n modulo 3, and

$$x_r = \begin{cases} 1 & \text{if } r = 0, \\ 2n(n+1)^{r-1} & \text{if } r \neq 0. \end{cases} \tag{3.2}$$

APPENDIX

The following combinatorial expression for odd-subscripted Lucas numbers emerges from a specialization of an expression for *generalized NSW numbers* (see [5], p. 288), a study of which is being undertaken by the author of this note. The interested reader might enjoy finding a proof for this expression:

$$L_{2n+1} = \frac{[1+(-1)^n](-1)^{n/2}}{2} + \sum_{j=0}^{\lfloor (n-1)/2 \rfloor} (-1)^j \binom{n-1-j}{j} 3^{n-1-2j} \frac{4n-5j}{n-2j} \quad (n \geq 0).$$

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A LAYMAN'S VIEW
of
Music of the Spheres

by Albert V. Carlin

When I contemplate
a page of symbols in mathematical array,
symmetrical and beautiful,
though I may not understand it all,
my mind rejoices
to think that here and now again
the human mind has come so far.
So far, to glimpse
the wondrous order
and balance of the Universe.

(Submitted by Herta T. Freitag, November 1997)

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