PRONIC LUCAS NUMBERS

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1. INTRODUCTION

An integer m is a pronic number if m is the product of two consecutive integers. We shall show that the only Lucas number which is a product of two consecutive integers is $L_0 = 2$.

The author has been informed by the referee that the results of this paper appeared recently in a Chinese journal (in Chinese) [2]; however, because of the relative inaccessibility of that article, the editor has accepted the referee's recommendation to publish the results in *The Fibonacci Quarterly*. The author has not yet seen the earlier publication, but understands that the proofs employ the same line of reasoning, although differing in details.

If $m = r(r+1)$, then $4m+1$ is a square. Our approach is to show that L_n, for $n > 0$, is not a pronic number by finding an integer $w(n)$ such that $4L_n + 1$ is a quadratic nonresidue modulo $w(n)$. It may be noted that if L_n is a pronic number, then L_n is two times a triangular number. Our interest in this problem was prompted by Ming Luo's very nice paper entitled "On Triangular Lucas Numbers," [2], and we employ an approach similar to that of Luo. We prove the following theorem.

Theorem: The Lucas number L_n is the product of two consecutive integers if and only if $n = 0$.

2. SOME IDENTITIES, SOME LEMMAS, AND THE PROOF

The Lucas numbers are defined by

$$L_0 = 2,\ L_1 = 1, \text{ and } L_n = L_{n-1} + L_{n-2}, \text{ for } n \geq 2,$$

and the recursive relation holds for n negative if $L_{-n} = (-1)^n L_n$.

Let n and m be any integers, and $\{F_n\}$ be the Fibonacci sequence. We require the following well-known identities:

$$L_n^2 = 5F_n^2 + 4(-1)^n; \tag{1}$$

$$L_{2n} = L_n^2 - 2(-1)^n; \tag{2}$$

$$2L_{m+n} = L_m L_n + 5F_m F_n; \tag{3}$$

$$L_{m+n} = L_m L_n - (-1)^n L_{m-n} = 5F_m F_n + (-1)^n L_{m-n}. \tag{4}$$

Our proof makes use of the periodicity of the sequence of Lucas numbers modulo an odd integer. It is well known [and easily shown using (4)] that, if t_k is an odd divisor of $5F_k$ and $n \equiv m \pmod{2k}$, then $L_n \equiv L_m \pmod{t_k}$. The reader may readily verify this fact using a table of Lucas numbers for these pairs used in the proofs: $(2k, t_k) = (8,3)$, $(4,5)$, $(16,7)$, $(10,11)$, $(20,25)$, $(50,101)$, $(44,89)$. $(22,199)$, $(88,43)$, and $(88,307)$.

Lemma 1: If L_n is pronic, then $n \equiv 0 \pmod{100}$.

Proof: Assume that $4L_n+1$ is a square. Then $4L_n+1$ is a quadratic residue modulo 11 and modulo 25. However, we find that $4L_n+1$ is a quadratic residue modulo 11 only if $n \equiv 0$, 1, or 5 (mod 10), i.e., $n \equiv 0$, 1, 5, 10, 11, or 15 (mod 20), and modulo 25 only if $n \equiv 0$, 4, 8, 12, or 16 (mod 20). Hence, $n \equiv 0$ (mod 20), so $n \equiv 0$, ± 20, ± 40 (mod 100). Since $L_{-n} = L_n$ for n even, it suffices to show that $4L_n+1$ is not a quadratic residue modulo 101 for $n \equiv 20$ and 40 (mod 100). We find that the Jacobi symbol

$$(4L_{20}+1|101) = (10|101) = -1,$$

and

$$(4L_{40}+1|101) = (89|101) = -1.$$

Lemma 2: If L_n is pronic, then $n \equiv 0$ (mod 88).

Proof: Assume $4L_n+1$ is a square. Then $4L_n+1$ is a quadratic residue modulo t_k, for $t_k = 3$, 5, and 7. However, the only integers n for which $4L_n+1$ is a quadratic residue modulo 3 *and* modulo 5 are $n \equiv 0$ and 5 (mod 8), and $4L_n+1$ is a quadratic nonresidue modulo 7 for $n \equiv 5$ and 13 (mod 16). Hence, $n \equiv 0$ (mod 8), so $n \equiv 0$, ± 8, ± 16, ± 24, ± 32, ± 40 (mod 88), and, as noted above, it suffices to show that $4L_n+1$ is not a quadratic residue for $n \equiv 8, 16, 24, 32,$ and 40 (mod 88). We find that $(4L_8+1|307) = (189|307)$, $(4L_{16}+1|199) = (73|199)$, $(4L_{24}+1|43) = (37|43)$, $(4L_{32}+1|43) = (3|43)$, and $(4L_{40}+1|89) = (29|89)$. Each Jacobi symbol equals -1, implying that L_n is pronic only if $n \equiv 0$ (mod 88).

Lemma 3: If $n = kg$, g odd, then

$$L_n \equiv \begin{cases} L_k \pmod{L_{2k}}, & \text{if } g \equiv 1, 3 \pmod 8, \\ -L_n \pmod{L_{2k}}, & \text{if } g \equiv 5, 7 \pmod 8. \end{cases}$$

Proof: By (4),

$$L_n = L_{k(g-2)}L_{2k} - (-1)^{2k} L_{k(g-4)} \equiv -L_{k(g-4)} \pmod{L_{2k}};$$

hence,

$$L_n = L_{kg} \equiv -L_{k(g-4)} \equiv +L_{k(g-8)} \equiv \cdots \equiv \pm L_{\pm k} = \pm L_k \pmod{L_{2k}}.$$

It is readily seen that the positive sign occurs if and only if $g = 1, 3$ (mod 8). □

In the following proof, we shall use the facts that L_m is odd if and only if $3 \nmid m$, and $L_{2^u m} \equiv -1$ (mod 8) if $u > 1$ and $3 \nmid m$.

Proof of the Theorem: If $n = 0$, $L_n = L_0 = 2$, a pronic number. Conversely, assume L_n is a pronic number. By Lemmas 1 and 2, $n = 2^u \cdot 5^2 \cdot 11t$, $u \geq 3$. Now, if $n = kg$, $2^u | k$, $3 \nmid k$, and g is odd, then, by Lemma 3,

$$\begin{aligned}(4L_n+1|L_{2k}) &= (\pm 4L_k + 1|L_{2k}) = \pm(4L_k \pm 1|L_{2k}) = (L_{2k}|4L_k \pm 1) \\ &= (L_k^2 - 2|4L_k \pm 1) = (16L_k^2 - 32|4L_k \pm 1) \\ &= ((4L_k+1)(4L_k-1) - 31|4L_k \pm 1) = (-31|4L_k \pm 1) \\ &= \pm(31|4L_k \pm 1) = (4L_k \pm 1|31).\end{aligned}$$

Case 1: $t \equiv 5$ or 7 (mod 8). Let $k = 2^u \cdot 5^2$ and $g = 11t \equiv 7$ or 5 (mod 8). By Lemma 3, $L_n \equiv -L_k \pmod{L_{2k}}$. Now, $L_{2\cdot 5^2} \equiv -1$ (mod 31) and, by induction [using (2)], $L_{2^u \cdot 5^2} \equiv -1$ (mod 31). Hence,

$$(4L_n + 1 | L_{2k}) = (4L_k - 1 | 31) = (-5 | 31) = -1.$$

Case 2: $t \equiv 1$ or 3 (mod 8). If $4 \nmid u$, let $k = 2^u$ and $g = 5^2 \cdot 11t \equiv 3$ or 1 (mod 8); if $4 | u$, let $k = 2^u \cdot 11$ and $g = 5^2 t \equiv 1$ or 3 (mod 8). By Lemma 3, $L_n \equiv L_k \pmod{L_{2k}}$. Using (2), we find that $4L_{2^u} + 1 \equiv 25, 13, -2, 3$ (mod 31) for $u \equiv 0, 1, 2, 3$ (mod 4), respectively. Then, if $4 \nmid u$,

$$(4L_n + 1 | L_{2k}) = (4L_{2^u} + 1 | 31) = (13 | 31), (-2 | 31), \text{ or } (3 | 31),$$

each of which equals -1.

Similarly, $4L_{2^u \cdot 11} + 1 \equiv -2, 3, 25, 13$ (mod 31) for $u \equiv 0, 1, 2, 3$ (mod 4), respectively; hence, for $4 | u$,

$$(4L_n + 1 | L_{2k}) = (4L_{2^u \cdot 11} + 1 | 31) = (-2 | 31) = -1. \quad \square$$

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REFERENCES

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