

ASYMPTOTIC ESTIMATION OF A SUM OF DIGITS

Harald Riede

University of Koblenz-Landau, Mathematisches Institut, Rheinau 1, 56075 Koblenz, Germany

(Submitted May 1996-Final Revision September 1996)

Let $s(k)$ denote the sum of the base 10 digits of $k \in \mathbf{N}$. For natural $x \geq 2$ and arbitrary fixed exponent $m \in \mathbf{N}$, it will be shown that

$$\frac{1}{x} \cdot \sum_{k=1}^{x-1} s(k)^m = \left(\frac{9}{2} \lg x\right)^m + O((\lg x)^{m-1}).$$

Here, "lg" denotes the base 10 log function. It is obvious that this formula can be generalized on arbitrary p -adic systems. The case $m=1$ has been treated in [1], $m=2$ in [2]; there the general case is exhibited as an *open problem*. The proof given now is based on induction.

I wish to thank Harald Scheid, University of Wuppertal, Germany, who drew my attention to certain unsolved arithmetical problems, the above among them.

1. THE ASSUMPTION

Let A_x for $x = 2, 3, \dots$ be the arithmetic function

$$A_x(m) = \sum_{k=0}^{x-1} s(k)^m, \quad m \in \mathbf{N}_0 (= \{0, 1, 2, \dots\}).$$

I denote the above assertion in the following manner,

$$A_x(i) = x \left(\frac{9}{2} \lg x\right)^i + d_i(x) \cdot x(\lg x)^{i-1}, \quad x \geq 2, \quad (1)$$

with certain bounded functions $d_i(x)$, i.e.,

$$|d_i(x)| \leq d_i \quad \text{for all } x, \quad (2)$$

and assume that it is valid for $i = 1, \dots, m-1$. The validity for $i = 1$ is guaranteed in [1] and the validity for $i = m$ will be deduced now in several steps.

2. A REDUCTION FORMULA FOR A_{10x}

The *binomial product* $B * C$ of two arithmetical functions is defined by

$$B * C(m) = \sum_{k=0}^m \binom{m}{k} B(k) \cdot C(m-k).$$

First, I will show that

$$A_{10x} = A_{10} * A_x. \quad (3)$$

$$A_{10x}(m) = \sum_{k=0}^{x-1} \sum_{i=0}^9 s(10k+i)^m = \sum_{k=0}^{x-1} \sum_{i=0}^9 (s(k)+i)^m = \sum_{k=0}^{x-1} \sum_{i=0}^9 \sum_{j=0}^m \binom{m}{j} s(k)^{m-j} i^j$$

$$\begin{aligned}
 &= \sum_k \sum_j \left(s(k)^{m-j} \binom{m}{j} \sum_{i=0}^9 i^j \right) = \sum_k \sum_j s(k)^{m-j} \binom{m}{j} A_{10}(j) = \sum_j \left(\binom{m}{j} A_{10}(j) \sum_k s(k)^{m-j} \right) \\
 &= \sum_j \binom{m}{j} A_{10}(j) \cdot A_x(m-j) = (A_{10} * A_x)(m).
 \end{aligned}$$

3. ESTIMATION OF THE REMAINDER

Let x have the decomposition $10y+z$ with $z < 10$. Suppose $R_x = A_x - A_{10y}$. In the case $z = 0$ we have $R_x = 0$, otherwise

$$R_x(m) = \sum_{i=0}^{z-1} s(10y+i)^m.$$

If $n+1$ denotes the number of digits of x , then

$$R_x(m) \leq z \cdot ((n+1) \cdot 9)^m \leq 9^{m+1} \cdot (n+1)^m.$$

Let $(a_n \dots a_0)$ be the decimal representation of x and $x_k = (a_n \dots a_k)$ (especially $x_0 = x$, $x_n = a_n$), then, in particular,

$$R_{x_k}(m) \leq 9^{m+1}(n-k+1)^m, \quad k = 0, 1, \dots, n. \tag{4}$$

4. A DECOMPOSITION OF $A_x(m)$

One can verify immediately that

$$\begin{aligned}
 A_x &= 10^n A_{x_n} + \sum_{k=1}^n (10^{k-1} A_{10x_k} - 10^k A_{x_k}) + \sum_{k=0}^{n-1} (10^k A_{x_k} - 10^k A_{10x_{k+1}}) \\
 &\stackrel{(3)}{=} 10^n A_{a_n} + \sum_{k=1}^n 10^{k-1} (A_{10} * A_{x_k} - 10 A_{x_k}) + \sum_{k=0}^{n-1} 10^k R_{x_k}, \\
 A_x(m) &= 10^n A_{a_n}(m) + \sum_{k=1}^n \left(10^{k-1} \sum_{i=1}^m \binom{m}{i} A_{10}(i) A_{x_k}(m-i) \right) + \sum_{k=0}^{n-1} 10^k R_{x_k}(m) \\
 &= \underbrace{10^n A_{a_n}(m)}_U + \underbrace{\sum_{i=1}^m \left(\underbrace{\binom{m}{i} \frac{A_{10}(i)}{10}}_{c_i} \underbrace{\sum_{k=1}^n 10^k A_{x_k}(m-i)}_{W_{m-i}} \right)}_W + \underbrace{\sum_{k=0}^{n-1} 10^k R_{x_k}(m)}_V
 \end{aligned}$$

The expressions U , V , and W shall be treated now one after another.

5. ESTIMATION OF U AND V

$U = 10^n A_{a_n}(m) = 10^n R_{a_n}(m) \stackrel{(4)}{\leq} 10^n \cdot 9^{m+1}$ and, since $10^n \leq x < 10^{n+1}$, we have $U = O(x)$. Furthermore,

$$V = \sum_{k=0}^{n-1} 10^k R_{x_k}(m) \stackrel{(4)}{\leq} 9^{m+1} \sum_{k=0}^{n-1} 10^k (n-k+1)^m.$$

Since the power series $\sum_k k^m z^k$ has radius of convergence 1, it is particularly convergent for $z = 1/10$; hence,

$$\sum_{k=0}^n 10^k (n-k+1)^m = 10^{n+1} \sum_{k=1}^{n+1} k^m \left(\frac{1}{10}\right)^k = O(x). \quad (5)$$

Thus, $V = O(x)$.

6. DECOMPOSITION AND ESTIMATION OF THE W_i

With respect to the assumption under induction, we obtain, for $i \leq m-1$,

$$\begin{aligned} W_i &= \sum_{i=1}^n 10^k A_{x_k}(i) = \sum_{k=1}^n 10^k \left(x_k \left(\frac{9}{2} \lg x_k\right)^i + d_i(x_k) \cdot x_k \cdot (\lg x_k)^{i-1} \right) \\ &= \left(\frac{9}{2}\right)^i \underbrace{\sum_{k=1}^n 10^k x_k (\lg x_k)^i}_{G_i} + \underbrace{\sum_{k=1}^n d_i(x_k) \cdot 10^k x_k (\lg x_k)^{i-1}}_{G_i^*}. \end{aligned}$$

Let $y_k = (a_k \dots a_0)$. Then $10^k x_k = \underbrace{(a_n \dots a_k 0 \dots 0)}_{n+1 \text{ digits}} = x - (a_{k-1} \dots a_0) = x - y_{k-1}$, so we have

$$G_i = \sum_{k=1}^n (x - y_{k-1}) (\lg x_k)^i = x \sum_{k=1}^n (\lg x_k)^i - \sum_{k=1}^n y_{k-1} (\lg x_k)^i.$$

The two sums herein shall now be estimated separately:

a) We have $(n-k)^i = (\lg 10^{n-k})^i \leq (\lg x_k)^i < (\lg 10^{n-k+1})^i = (n-k+1)^i$; hence,

$$\sum_{k=1}^n (n-k)^i \leq \sum_k (\lg x_k)^i < \sum_{k=1}^n (n-k+1)^i = n^i + \sum_{k=1}^n (n-k)^i.$$

Since

$$\sum_{k=1}^n (n-k)^i = \sum_{k=1}^{n-1} k^i = \frac{n^{i+1}}{i+1} + O(n^i),$$

we see that, for arbitrary $i \in \mathbb{N}$,

$$\sum_{k=1}^n (\lg x_k)^i = \frac{n^{i+1}}{i+1} + O(n^i).$$

b) $\sum_{k=1}^n y_{k-1} (\lg x_k)^i \leq \sum_k 10^k (\lg 10^{n-k+1})^i = \sum_k 10^k (n-k+1)^i \stackrel{(5)}{=} O(x)$.

Putting the two parts together, we have

$$G_i = x \cdot \frac{n^{i+1}}{i+1} + O(x \cdot n^i),$$

particularly with respect to (2): $|G_i^*| \leq d_i G_{i-1} = O(x \cdot n^i)$; therefore,

$$W_i = \left(\frac{9}{2}\right)^i G_i + G_i^* = \left(\frac{9}{2}\right)^i x \cdot \frac{n^{i+1}}{i+1} + O(x \cdot n^i) \text{ for all } i \leq m-1.$$

Now it is easily seen that

$$\begin{aligned}
 W &= \sum_{i=1}^m \binom{m}{i} \frac{A_{10}(i)}{10} W_{m-i} = m \frac{A_{10}(1)}{10} W_{m-1} + O(x \cdot n^{m-1}) \\
 &= m \cdot \frac{9}{2} \cdot \left(\frac{9}{2}\right)^{m-1} x \cdot \frac{n^m}{m} + O(x \cdot n^{m-1}) = \left(\frac{9}{2}\right)^m x \cdot n^m + O(x \cdot n^{m-1}).
 \end{aligned}$$

And, finally,

$$A_x(m) = \left(\frac{9}{2}\right)^m x \cdot n^m + O(x \cdot n^{m-1}).$$

From this, the initial assertion is deduced immediately.

Often a solved problem procreates a new problem. Here is an open question: Does the given asymptotic estimation hold even for arbitrary *real* $m \geq 1$? The reader is invited to prove or disprove this result.

REFERENCES

1. P. H. Cheo & S. C. Yien. "A Problem on the k -adic Representation of Positive integers." *Acta Math. Sinica* 5 (1955):433-48.
2. R. E. Kennedy & C. N. Cooper. "An Extension of a Theorem by Cheo and Yien Concerning Digital Sums." *The Fibonacci Quarterly* 29.2 (1991):145-49.

AMS Classification Numbers: 10J06, 10L99, 10A99



Author and Title Index

The AUTHOR, TITLE, KEY-WORD, ELEMENTARY PROBLEMS, and ADVANCED PROBLEMS indices for the first 30 volumes of *The Fibonacci Quarterly* have been completed by Dr. Charles K. Cook. Publication of the completed indices is on a 3.5-inch, high density disk. The price for a copyrighted version of the disk will be \$40.00 plus postage for non-subscribers, while subscribers to *The Fibonacci Quarterly* need only pay \$20.00 plus postage. For additional information, or to order a disk copy of the indices, write to:

PROFESSOR CHARLES K. COOK
 DEPARTMENT OF MATHEMATICS
 UNIVERSITY OF SOUTH CAROLINA AT SUMTER
 1 LOUISE CIRCLE
 SUMTER, SC 29150

The indices have been compiled using WORDPERFECT. Should you wish to order a copy of the indices for another wordprocessor or for a non-compatible IBM machine, please explain your situation to Dr. Cook when you place your order and he will try to accommodate you. **DO NOT SEND PAYMENT WITH YOUR ORDER.** You will be billed for the indices and postage by Dr. Cook when he sends you the disk. A star is used in the indices to indicate unsolved problems. Furthermore, Dr. Cook is working on a SUBJECT index and will also be classifying all articles by use of the AMS Classification Scheme. Those who purchase the indices will be given one free update of all indices when the SUBJECT index and the AMS Classification of all articles published in *The Fibonacci Quarterly* are completed.