A CLASS OF SEQUENCES AND THE AITKEN TRANSFORMATION

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1. INTRODUCTION

In the notation of Horadam [1], let $W_n = W_n(a,b;\, p,q)$, where

$$W_n = pW_{n-1} - qW_{n-2} \quad (n \geq 2), \quad W_0 = a, \ W_1 = b. \tag{1.1}$$

If α and β are assumed distinct, then the roots of $\lambda^2 - p\lambda + q = 0$ have the Binet form

$$W_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta} \tag{1.2}$$

in which $A = b - a\beta$ and $B = b - a\alpha$.

The n^{th} terms of the Fibonacci and Lucas sequences are:

$$F_n = W_n(0,1;\, 1,-1); \quad L_n = W_n(2,1;\, 1,-1). \tag{1.3}$$

As usual, we write

$$U_n = W_n(0,1;\, p,q) = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad V_n = W_n(2,p;\, p,q) = \alpha^n + \beta^n, \tag{1.4}$$

where $\{U_n\}$ and $\{V_n\}$ are the fundamental and primordial sequences, respectively, generated by (1.3). These sequences have been studied extensively, particularly by Lucas [3] and Horadam [1]. Throughout this paper, d is a natural number.

Define the Aitken transformation by

$$A(x, x', x'') = (xx'' - x'^2)/(x - 2x' + x''), \tag{1.5}$$

where the denominator is assumed to be nonzero.

In 1984, Phillips discovered the following relation between ratios of Fibonacci numbers and the Aitken transformation,

$$A(r_{n-t}, r_n, r_{n+t}) = r_{2n}, \tag{1.6}$$

where $r_n = F_{n+1}/F_n$. An account of this work is also given by Vajda in [3]. McCabe and Phillips [5] generalized this to show that (1.6) holds when $r_n = U_{n+1}/U_n$, and Muskat [7] showed that (1.6) holds for $r_n = U_{n+d}/U_n$. Jamieson [6] obtained the generalization

$$A(W_{i-t}^{(k)}, W_i^{(k)}, W_{i+t}^{(k)}) = \begin{cases} W_{2i}^{(2k)}, & 2k < p, \\ W_{2i}^{(2k-p)}, & 2k \geq p, \end{cases} \tag{1.7}$$

where $W_i^{(k)} = F_{p(i+1)-k}/F_{pi-k}$, $0 \leq k \leq p-1$.

The purpose of this paper is to establish a further generalization of these results.

2. THE MAIN RESULTS

First we introduce a new class of more general sequences that has not appeared previously in the literature.

Definition: The generalized Fibonacci sequence (GF-Sequence) is defined by

$$W_{n,d}^{(k)}(a,b;\,p,q) = \frac{A^k\alpha^{nk+d} - B^k\beta^{nk+d}}{\alpha-\beta}. \tag{2.1}$$

Thus, we have $F_n = W_{n,0}^{(1)}(0,1;\,1,-1)$, $U_n = W_{n,0}^{(1)}(0,1;\,p,q)$, and $W_n = W_{n,0}^{(1)}(a,b;\,p,q)$, and the GF-sequence $W_{n,d}^{(k)}(a,b;\,p,q)$ is seen to be an extension of these sequences.

We write $W_{n,d}^{(k)}$ for $W_{n,d}^{(k)}(a,b;\,p,q)$ and note that this sequence satisfies the recurrence relation

$$W_{n+1,d}^{(k)} = (\alpha^k+\beta^k)W_{n,d}^{(k)} - \alpha^k\beta^k W_{n-1,d}^{(k)},$$

which has characteristic equation with roots α^k and β^k and generating function

$$\sum_{n=0}^{\infty} W_{n,d}^{(k)}t^n = \frac{A^k\alpha^d - B^k\beta^d - (A^k\alpha^d\beta^k - B^k\alpha^k\beta^d)t}{(\alpha-\beta)(1-(\alpha^k+\beta^k)t+\alpha^k\beta^k t^2)} = \frac{W_{0,d}^{(k)} - q^k W_{-1,d}^{(k)}t}{1-V_k t+q^k t^2}.$$

Introducing such a class of generalized Fibonacci sequences $W_{n,d}^{(k)}$, we can find a nice property between the appropriate ratios involving this sequence and Aitken acceleration.

If $W_{n,0}^{(k)} \neq 0$, we define the ratio

$$R_n^{(k)} = W_{n,d}^{(k)} / W_{n,0}^{(k)} \tag{2.2}$$

and state the main result of this paper.

Theorem:

$$A(R_{n-t}^{(k)},\, R_n^{(k)},\, R_{n+t}^{(k)}) = R_n^{(2k)}. \tag{2.3}$$

3. LEMMA

For the proof of the Theorem, we introduce the following lemma.

Lemma:

(a) $$W_{n+t,d}^{(k)}W_{n-t,d}^{(k)} - (W_{n,d}^{(k)})^2 = -A^kB^kq^{(n-t)k+d}(U_{kt})^2, \tag{3.1}$$

(b) $$W_{n,0}^{(k)}W_{n-t,d}^{(k)} - W_{n,d}^{(k)}W_{n-t,0}^{(k)} = A^kB^kq^{(n-t)k}U_dU_{kt}, \tag{3.2}$$

(c) $$W_{n,d}^{(k)}W_{n+t,0}^{(k)} - W_{n,0}^{(k)}W_{n+t,d}^{(k)} = A^kB^kq^{nk}U_dU_{kt}, \tag{3.3}$$

(d) $$(W_{n,d}^{(k)})^2 - q^d(W_{n,0}^{(k)})^2 = U_dW_{n,d}^{(2k)}, \tag{3.4}$$

(e) $$W_{n+t,0}^{(k)} - q^{kt}W_{n-t,0}^{(k)} = U_{kt}(A^k\alpha^{nk} + B^k\beta^{nk}). \tag{3.5}$$

Proof: We prove only part (a) because the proofs of (b)-(e) are similar. Using the definition of $W_{n,d}^{(k)}$, we have

$$W_{n+t,d}^{(k)}W_{n-t,d}^{(k)}-(W_{n,d}^{(k)})^2$$
$$=\frac{A^k\alpha^{(n+t)k+d}-B^k\beta^{(n+t)k+d}}{\alpha-\beta}\,\frac{A^k\alpha^{(n-t)k+d}-B^k\beta^{(n-t)k+d}}{\alpha-\beta}-\left(\frac{A^k\alpha^{nk+d}-B^k\beta^{nk+d}}{\alpha-\beta}\right)^2$$
$$=-A^kB^kq^{(n-t)k+d}\left(\frac{\alpha^{tk}-\beta^{tk}}{\alpha-\beta}\right)^2=-A^kB^kq^{(n-t)k+d}(U_{kt})^2$$

and the proof of (a) is complete.

4. PROOF OF THE THEOREM

Using (1.5) and (2.2), we may write

$$A(R_{n-t}^{(k)},R_n^{(k)},R_{n+t}^{(k)})=\frac{R_{n-t}^{(k)}R_{n+t}^{(k)}-(R_n^{(k)})^2}{R_{n-t}^{(k)}-2R_n^{(k)}+R_{n+t}^{(k)}}=\frac{\dfrac{W_{n-t,d}^{(k)}W_{n+t,d}^{(k)}}{W_{n-t,0}^{(k)}W_{n+t,0}^{(k)}}-\left(\dfrac{W_{n,d}^{(k)}}{W_{n,0}^{(k)}}\right)^2}{\dfrac{W_{n-t,d}^{(k)}}{W_{n-t,0}^{(k)}}-2\dfrac{W_{n,d}^{(k)}}{W_{n,0}^{(k)}}+\dfrac{W_{n+t,d}^{(k)}}{W_{n+t,0}^{(k)}}}$$

$$=\frac{(W_{n,0}^{(k)})^2W_{n-t,d}^{(k)}W_{n+t,d}^{(k)}-(W_{n,d}^{(k)})^2W_{n-t,0}^{(k)}W_{n+t,0}^{(k)}}{(W_{n,0}^{(k)})^2W_{n-t,d}^{(k)}W_{n+t,0}^{(k)}-2W_{n,0}^{(k)}W_{n,d}^{(k)}W_{n-t,0}^{(k)}W_{n+t,0}^{(k)}+(W_{n,0}^{(k)})^2W_{n+t,d}^{(k)}W_{n-t,0}^{(k)}}$$

$$=\frac{(W_{n,0}^{(k)})^2(W_{n-t,d}^{(k)}W_{n+t,d}^{(k)}-(W_{n,d}^{(k)})^2)-(W_{n,d}^{(k)})^2(W_{n-t,0}^{(k)}W_{n+t,0}^{(k)}-(W_{n,0}^{(k)})^2)}{W_{n,0}^{(k)}[W_{n+t,0}^{(k)}(W_{n,0}^{(k)}W_{n-t,d}^{(k)}-W_{n,d}^{(k)}W_{n-t,0}^{(k)})-W_{n-t,0}^{(k)}(W_{n,d}^{(k)}W_{n+t,0}^{(k)}-W_{n,0}^{(k)}W_{n+t,d}^{(k)})]}$$

$$=\frac{(W_{n,0}^{(k)})^2(-A^kB^kq^{(n-t)k+d})U_{kt}^2-(W_{n,d}^{(k)})^2(-A^kB^kq^{(n-t)k})U_{kt}^2}{W_{n,0}^{(k)}[W_{n+t,0}^{(k)}A^kB^kq^{(n-t)k}U_{kt}U_d-W_{n-t,0}^{(k)}A^kB^kq^{nk}U_{kt}U_d]}$$

$$=\frac{U_{kt}[(W_{n,d}^{(k)})^2-q^d(W_{n,0}^{(k)})^2]}{W_{n,0}^{(k)}U_d[W_{n+t,0}^{(k)}-q^{tk}W_{n-t,0}^{(k)}]},\text{ by (3.1) and (3.2),}$$

$$=\frac{U_{kt}U_dW_{n,d}^{(2k)}}{W_{n,0}^{(k)}U_dU_{kt}(A^k\alpha^{nk}+B^k\beta^{nk})},\text{ by (3.3), (3.4), and (3.5),}$$

$$=\frac{W_{n,d}^{(2k)}}{W_{n,0}^{(2k)}}=R_n^{(2k)}.$$

This completes the proof of the Theorem.

5. REMARK

There is a major difference between the result of this paper and those of other papers on this topic. In this paper, when the Aitken transformation is applied to the three numbers, $R_{n-t}^{(k)}$, $R_n^{(k)}$, and $R_{n+t}^{(k)}$, we obtain a doubling of k, giving $R_n^{(2k)}$. This contrasts with the results of all the other authors quoted, such as the relation $A(r_{n-t},r_n,r_{n+t})=r_{2n}$, where it is n that is doubled.

But, when $k = 1, a = 0$, and $b = 1$, we have $R_n^{(2)} = U_{2n+d} / U_{2n} = r_{2n}$. Thus, the result of this paper may be regarded as a further generalization of the former results.

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