ON THE INTEGRITY OF CERTAIN INFINITE SERIES

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1. INTRODUCTION

We consider the second-order recurring relation

$$W_0 = a,\ W_1 = b,\ W_n = PW_{n-1} - QW_{n-2} \quad (n \ge 2), \tag{1.1}$$

where a, b, P, and Q are integers, with $P > 0$, $Q \ne 0$, and $\Delta = P^2 - 4Q > 0$. Particular cases of $\{W_n\}$ are the sequences $\{U_n\} = \{U_n(P, Q)\}$ of Fibonacci and $\{V_n\} = \{V_n(P, Q)\}$ of Lucas defined by $U_0 = 0, U_1 = 1$ and $V_0 = 2, V_1 = P$, respectively. It is well known that

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad V_n = \alpha^n + \beta^n, \tag{1.2}$$

where

$$\alpha = \frac{P + \sqrt{\Delta}}{2} \quad \text{and} \quad \beta = \frac{P - \sqrt{\Delta}}{2}, \tag{1.3}$$

so that $\alpha + \beta = P$ and $\alpha\beta = Q$.

Since $P > 0$, notice that $\alpha > 1$, $\alpha > |\beta|$, so that $U_n > 0$ $(n \ge 1)$, $V_n > 0$ $(n \ge 0)$.

Recently, several papers ([2], [3], and [6]) have been devoted to the study of the infinite sum

$$S_U(x) = S_U(x; P, Q) = \sum_{n=0}^{\infty} \frac{U_n}{x^n}. \tag{1.4}$$

The main known results can be summarized as follows.

Theorem 1:

(i) If $Q = -1$, the rational values of $x = r/s$ for which $S_U(x)$ is an integer are given by

$$x = \frac{U_{2n+1}}{U_{2n}} \quad (n = 1, 2, \ldots), \tag{1.5}$$

and the corresponding value of S_U is given by

$$S_U(x) = U_{2n}U_{2n+1}. \tag{1.6}$$

(ii) If $Q = 1$ and $P \ge 3$, the rational values of $x = r/s$ for which $S_U(x)$ is an integer are given by

$$x = \frac{U_{n+1}}{U_n} \quad (n = 1, 2, \ldots), \tag{1.7}$$

and the corresponding value of S_U is given by

$$S_U(x) = U_nU_{n+1}. \tag{1.8}$$

The aim of this paper is to extend the above result to the infinite sum

$$S_V(x) = S_V(x; P, Q) = \sum_{n=0}^{\infty} \frac{V_n}{x^n}, \quad \text{where } Q = \pm 1. \tag{1.9}$$

Using the Binet forms (1.2) and the geometric series formula, we get the closed-form expression

$$S_V(x)=\frac{x(2x-P)}{x^2-Px+Q}, \quad |x|>\alpha. \tag{1.10}$$

Remark 1.1: We have assumed that $P>0$. Actually, it is well known that

$$U_n(-P,Q)=(-1)^{n-1}U_n(P,Q) \text{ and } V_n(-P,Q)=(-1)^nV_n(P,Q).$$

From this, we get

$$S_U(x;\,-P,Q)=-S_U(-x;\,P,Q) \text{ and } S_V(x;\,-P,Q)=S_V(-x;\,P,Q).$$

Thus, the case $P<0$ cannot give really new results.

Remark 1.2: It is clear by (1.9) that $S_V(x)>V_0=2$ for $x>\alpha$, since $V_n>0$ for every $n\geq 0$.

In what follows, we shall make use of the well-known identities:

$$V_n+PU_n=2U_{n+1}; \tag{1.11}$$
$$\Delta U_n+PV_n=2V_{n+1}; \tag{1.12}$$
$$U_{2n}=U_nV_n; \tag{1.13}$$
$$V_{2n}+2Q^n=V_n^2; \tag{1.14}$$
$$V_{2n}-2Q^n=\Delta U_n^2; \tag{1.15}$$
$$V_n^2-\Delta U_n^2=4Q^n; \tag{1.16}$$
$$U_{2n+1}=U_{n+1}^2-QU_n^2. \tag{1.17}$$

All of these identities can be proved by using the Binet forms (1.2).

2. MAIN RESULTS

Theorem 2: If $Q=\pm 1$, there do not exist negative rational values of x such that $S_V(x)$ is an integer, except when $Q=-1$ and $P=1$. In this case, the only solution is given by $x=-2$, with $S_V(-2)=2$.

Remark 2.1: Since $V_n(1,-1)=L_n$ (the n^{th} Lucas number), we see by Theorem 2 that

$$\sum_{n=0}^{\infty}\frac{(-1)^nL_n}{2^n}=2. \tag{2.1}$$

Theorem 3: If $Q=-1$, the positive rational values of x for which $S_V(x)$ is integral are given by

$$x=\frac{U_{2n+1}}{U_{2n}} \quad (n=1,2,\ldots) \tag{2.2}$$

and

$$x=\frac{V_{2n+2}}{V_{2n+1}} \quad (n=0,1,\ldots). \tag{2.3}$$

The corresponding values of $S_V(x)$ are given by

$$S_V(U_{2n+1}/U_{2n}) = U_{2n+1}V_{2n} \tag{2.4}$$

and

$$S_V(V_{2n+2}/V_{2n+1}) = U_{2n+1}V_{2n+2}. \tag{2.5}$$

Theorem 4: If $Q=1$ and $P \geq 3$, the positive rational values of x for which $S_V(x)$ is integral are given by

$$x = \frac{U_{n+1}}{U_n} \quad (n = 1, 2, \ldots) \tag{2.6}$$

and

$$x = \frac{X_{n+1}}{X_n} \quad (n = 0, 1, \ldots), \tag{2.7}$$

where $X_n = U_{n+1} + U_n$.

The corresponding values of $S_V(x)$ are given by

$$S_V(U_{n+1}/U_n) = U_{n+1}V_n \tag{2.8}$$

and

$$S_V(X_{n+1}/X_n) = X_{n+1}(U_{n+1} - U_n). \tag{2.9}$$

3. PROOF OF THEOREM 2

Consider the function ϕ defined by

$$\phi(x) = \frac{x(2x-P)}{x^2 - Px + Q}, \quad x \neq \alpha \text{ and } x \neq \beta. \tag{3.1}$$

From (1.10) it is clear that $\phi(x) = S_V(x)$ when $|x| > \alpha$, and one can see immediately that

$$\lim_{x \to -\infty} \phi(x) = 2 \quad \text{and} \quad \phi(-\alpha) = 1 + \frac{\sqrt{\Delta}}{2P} > 1. \tag{3.2}$$

Assuming first that $Q = 1$, we see that ϕ is decreasing on $]-\infty, \beta]$ and thus on $]-\infty, -\alpha]$ (recall that $-\alpha < \beta$, since $P > 0$). By (3.2), it is clear that there does not exist a number $x < -\alpha$ with $\phi(x)$ an integer.

Assuming now that $Q = -1$, we see that ϕ is decreasing on $]-\infty, \gamma]$ with $\gamma = \frac{-2-\sqrt{\Delta}}{P}$, and it is not hard to prove that $\phi(\gamma) = 1 + \frac{2}{\sqrt{\Delta}} > 1$. If $P \geq 2$, one verifies that $-\alpha < \gamma$, and the same conclusion follows. On the other hand, if $P = 1$, we have $\gamma = -2 - \sqrt{5} = -4.2\ldots$, $\phi(\gamma) = 1 + \frac{2}{\sqrt{5}} = 1.8\ldots$, $-\alpha = -1.6\ldots$, $\phi(-\alpha) = 1 + \frac{\sqrt{5}}{2} = 2.1\ldots$, and that ϕ is increasing on $[\gamma, -\alpha]$. Thus, 2 is the only integer value of ϕ within this interval, and it is immediate that $\phi(-1) = 2$ This completes the proof.

To prove Theorems 3 and 4, we need some further mathematical tools. These will be discussed in Sections 4 and 6.

4. A PELL EQUATION

In this section we shall suppose that $Q = \pm 1$. Let $x = r/s > \alpha$, where r and s are positive integers with $\gcd(r, s) = 1$. We see by (1.10) that

$$S_V(r/s) = rk, \tag{4.1}$$

where

$$k = \frac{(2r - Ps)}{r^2 - Prs + Qs^2}. \tag{4.2}$$

It is clear that $k > 0$, since $S_V(r/s) > 0$ by Remark 1.2. We also see that $\gcd(r, r^2 - Prs + Qs^2) = 1$, since $Q = \pm 1$ and $\gcd(r, s) = 1$. From this fact, we see that $S_V(r/s)$ is an integer if and only if

$$k = \frac{2r - Ps}{r^2 - Prs + Qs^2} = \frac{4(2r - Ps)}{(2r - Ps)^2 - \Delta s^2}$$

is an integer. Putting $z = 2r - Ps$ for notational convenience, we get the second-degree equation in the unknown z

$$\frac{4z}{z^2 - \Delta s^2} = k, \tag{4.3}$$

which can be written as

$$kz^2 - 4z - k\Delta s^2 = 0. \tag{4.4}$$

Notice that $z > s\sqrt{\Delta} > 0$, since $r/s > \alpha = \frac{P}{2} + \frac{\sqrt{\Delta}}{2}$. The only positive root of (4.4) is given by

$$z = \frac{2 + \sqrt{d}}{k}, \tag{4.5}$$

where $d = 4 + \Delta(ks)^2$. For z to be an integer, the inequality

$$4 + \Delta(ks)^2 = y^2 \quad (y = 0, 1, \dots) \tag{4.6}$$

must hold. Observing that $\Delta = P^2 \pm 4$ is never a square, it follows by (1.16) and the theory of Pell equation (see, e.g., [5] and [7]) that the solutions of (4.6) in the unknown y and ks are given by

$$y = V_{2n},\ ks = U_{2n} \ \ (n \ge 0),\ \text{if } Q = -1, \tag{4.7}$$

and by

$$y = V_n,\ ks = U_n \ \ (n \ge 0),\ \text{if } Q = 1. \tag{4.8}$$

In our problem we can suppose that $n \ge 1$, since $ks > 0$, and we have to consider the two cases ($Q = 1$ and $Q = -1$), separately.

5. PROOF OF THEOREM 3

In this section we suppose that $Q = -1$. Assuming that $S_V(r/s)$ is an integer, we see by (4.7) that $ks = U_{2n}$ and $\sqrt{d} = y = V_{2n}$ for $n \ge 1$. It follows by (4.5), (1.13), (1.14), and (1.15) that

$$z = s\frac{2 + \sqrt{d}}{ks} = s\frac{2 + V_{2n}}{U_{2n}} \tag{5.1}$$

$$= \begin{cases} s\dfrac{V_n^2}{U_{2n}} = s\dfrac{V_n}{U_n}, & n \ge 2 \text{ even}, \\[2ex] s\dfrac{\Delta U_n^2}{U_{2n}} = s\dfrac{\Delta U_n}{V_n}, & n \text{ odd}. \end{cases} \tag{5.2}$$

On the other hand, recalling that $z = 2r - Ps$ and using (1.11) and (1.12), we see that

$$r = \frac{z+Ps}{2} = \begin{cases} s\dfrac{V_n + PU_n}{2U_n} = s\dfrac{U_{n+1}}{U_n}, & n \geq 2 \text{ even}, \\ s\dfrac{\Delta U_n + PV_n}{2V_n} = s\dfrac{V_{n+1}}{V_n}, & n \text{ odd}. \end{cases} \tag{5.3}$$

Finally, we get

$$x = r/s = \begin{cases} \dfrac{U_{n+1}}{U_n}, & n \text{ even and positive}, \\ \dfrac{V_{n+1}}{V_n}, & n \text{ odd}, \end{cases} \qquad \text{[cf. (2.2) and (2.3)]}.$$

To prove the second part of the theorem, notice first that $\frac{U_{n+1}}{U_n} > \alpha$ $(n \geq 2$ even) and $\frac{V_{n+1}}{V_n} > \alpha$ (n odd), since $Q = -1$. From (4.1), we see that

$$S_V(r/s) = rk = \frac{r}{s}ks. \tag{5.4}$$

Putting $r/s = U_{n+1}/U_n$ $(n \geq 2$ even) in (5.4) and using (4.7) and (1.13), we get

$$S_V(U_{n+1}/U_n) = \frac{U_{n+1}}{U_n}U_{2n} = U_{n+1}V_n \quad (n \geq 2 \text{ even}) \quad \text{[cf. (2.4)]}.$$

Now, putting $r/s = V_{n+1}/V_n$ (n odd) in (5.4), we obtain

$$S_V(V_{n+1}/V_n) = \frac{V_{n+1}}{V_n}U_{2n} = V_{n+1}U_n \quad (n \text{ odd}) \quad \text{[cf. (2.5)]}.$$

This completes the proof of Theorem 3. For the proof of Theorem 4, we need some results on the Fibonacci and Lucas numbers with real subscripts. These will be discussed in Section 6.

6. FIBONACCI AND LUCAS FUNCTIONS

Several definitions of Fibonacci and Lucas numbers with real subscripts are available in the literature (see, e.g., [1] and [4]) for the case in which $P = -Q = 1$.

Let us suppose here that $Q = 1$ and $P \geq 3$. Thus, α and β as defined by (1.3) are positive quantities and we can define, for every real number x, the real quantities

$$U_x = \frac{\alpha^x - \beta^x}{\alpha - \beta} \quad \text{and} \quad V_x = \alpha^x + \beta^x. \tag{6.1}$$

Using (6.1), the following identities can readily be found:

$$U_x = PU_{x-1} - U_{x-2} \quad \text{and} \quad V_x = PV_{x-1} - V_{x-2}, \tag{6.2}$$

$$U_x = U_{x/2}V_{x/2}, \tag{6.3}$$

$$V_x + 2 = V_{x/2}^2, \tag{6.4}$$

$$V_x + PU_x = 2U_{x+1}, \tag{6.5}$$

$$U_{x+y} + U_{x-y} = U_xV_y, \text{ for every } x \text{ and every } y. \tag{6.6}$$

The sequences $Y_n = U_{n+1/2}$ will be of particular interest for our purposes. Putting $x = n+1/2$ and $y = 1/2$ in (6.6), we get

$$Y_n = V_{1/2}^{-1} X_n, \tag{6.7}$$

where $X_n = U_{n+1} + U_n$, as specified in Theorem 4.

7. PROOF OF THEOREM 4

In this section we suppose that $Q = 1$ and $P \geq 3$. Assuming by (4.8) that $ks = U_n$ and $\sqrt{d} = y = V_n$ $(n \geq 1)$, it follows by (6.3) and (6.4) that

$$z = s\frac{2+\sqrt{d}}{ks} = s\frac{2+V_n}{U_n} = s\frac{V_{n/2}^2}{U_n} = s\frac{V_{n/2}}{U_{n/2}}. \tag{6.8}$$

Now by (6.5) we get

$$r = \frac{z+Ps}{2} = s\frac{V_{n/2}+PU_{n/2}}{2U_{n/2}} = s\frac{U_{n/2+1}}{U_{n/2}}. \tag{6.9}$$

Hence, letting $n = 2m$ and $n = 2m+1$ in (6.9) and using (6.7) in the latter case, one obtains

$$x = r/s = \begin{cases} \dfrac{U_{m+1}}{U_m}, & m > 0, \\ \dfrac{Y_{m+1}}{Y_m} = \dfrac{X_{m+1}}{X_m}, & m \geq 0, \end{cases} \quad \text{[cf. (2.6) and (2.7)]}.$$

As for the second part of the theorem, we see that $U_{m+1}/U_m > \alpha$ $(m > 0)$ and $X_{m+1}/X_m > \alpha$ $(m \geq 0)$, since $Q = 1$. Putting $r/s = U_{m+1}/U_m$ in (5.4) and using (4.8) (with $n = 2m$), we get

$$S_V(U_{m+1}/U_m) = \frac{U_{m+1}}{U_m} U_{2m} = U_{m+1}V_m \quad \text{[cf. (2.8)]}.$$

Finally, from (4.8) (with $n = 2m+1$) and (1.17), we get

$$\begin{aligned} S_v(X_{m+1}/X_m) &= \frac{X_{m+1}}{X_m} U_{2m+1} = \frac{X_{m+1}}{X_m}(U_{m+1}^2 - U_m^2) \\ &= X_{m+1}(U_{m+1} - U_m) \quad \text{[cf. (2.9)]}, \end{aligned}$$

and the proof is complete.

Concluding Remark: From Theorems 3 and 4, one can study the integrity of the infinite sum

$$T_k(x) = \sum_{n=0}^{\infty} \frac{V_{kn}}{x^n}, \quad k > 0.$$

This investigation might be the aim of a future work.

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