PALINDROMIC SEQUENCES FROM IRRATIONAL NUMBERS

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In this paper, a *palindrome* is a finite sequence $(x(1), x(2), \dots, x(n))$ of numbers satisfying $(x(1), x(2), \dots, x(n)) = (x(n), x(n-1), \dots, x(1))$. Of course, an infinite sequence cannot be a palindrome—however, we shall call an infinite sequence $x = (x(1), x(2), \dots)$ a *palindromic sequence* if for every N there exists $n > N$ such that the finite sequence $(x(1), x(2), \dots, x(n))$ is a palindrome. If α is an irrational number, then the sequence Δ defined by $\Delta(n) = \lfloor n\alpha \rfloor - \lfloor n\alpha - \alpha \rfloor$ is, we shall show, palindromic.

Lemma: Suppose $\sigma = (\sigma(0), \sigma(1), \sigma(2), \dots)$ is a sequence of numbers, and $\sigma(0) = 0$. Let Δ be the sequence defined by $\Delta(n) = \sigma(n) - \sigma(n-1)$ for $n = 1, 2, 3, \dots$. Then Δ is a palindromic sequence if and only if there are infinitely many n for which the equations

$$F_{n,k}\colon\ \sigma(k) + \sigma(n-k) = \sigma(n) \tag{1}$$

hold for $k = 1, 2, \dots, n$.

Proof: Equations $F_{n,k}$ and $F_{n,k-1}$ yield $\sigma(k) + \sigma(n-k) = \sigma(n) = \sigma(k-1) + \sigma(n-k+1)$, so that the equations

$$E_{n,k}\colon\ \sigma(k) - \sigma(k-1) = \sigma(n-k+1) - \sigma(n-k) \tag{2}$$

or

$$\Delta(k) = \Delta(n-k+1)$$

follow, for $k = 1, 2, \dots, n$. Thus, if the n equations (1) hold for infinitely many n, then Δ is a palindromic sequence.

For the converse, suppose n is a positive integer for which the equations $E_{n,k}$ in (2) hold. The equations $E_{n,1}$, $E_{n,1} + E_{n,2}$, $E_{n,1} + E_{n,2} + E_{n,2}$, $\dots$, $E_{n,1} + E_{n,2} + \dots + E_{n,n}$ readily reduce to the equations $F_{n,k}$. Thus, if Δ is palindromic, then the equations $F_{n,k}$, for $k = 1, 2, \dots, n$, hold for infinitely many n. □

To see how a positive irrational number α can be used to generate palindromic sequences, we recall certain customary notations from the theory of continued fractions. Suppose α has continued fraction $[\![a_0, a_1, a_2, \dots]\!]$, and let $p_{-2} = 0, p_{-1} = 1, p_i = a_i p_{i-1} + p_{i-2}$ and $q_{-2} = 1, q_{-1} = 0$, $q_i = a_i q_{i-1} + q_{i-2}$ for $i \geq 0$. The *principal convergents of* α are the rational numbers p_i / q_i for $i \geq 0$. Now, for all nonnegative integers i and j, define $p_{i,j} = jp_{i+1} + p_i$ and $q_{i,j} = jq_{i+1} + q_i$. The fractions

$$\frac{p_{i,j}}{q_{i,j}} = \frac{jp_{i+1} + p_i}{jq_{i+1} + q_i}, \qquad 1 \leq j \leq a_{i+2} - 1, \tag{3}$$

are the i^{th} *intermediate convergents of* α. As proved in [2, p. 16],

$$\dots < \frac{p_i}{q_i} < \dots < \frac{p_{i,j}}{q_{i,j}} < \frac{p_{i,j+1}}{q_{i,j+1}} < \dots < \frac{p_{i+2}}{q_{i+2}} < \dots \quad \text{if } i \text{ is even,} \tag{4}$$

$$\cdots > \frac{p_i}{q_i} > \cdots > \frac{p_{i,j}}{q_{i,j}} > \frac{p_{i,j+1}}{q_{i,j+1}} > \cdots > \frac{p_{i+2}}{q_{i+2}} > \cdots \text{ if } i \text{ is odd,} \tag{5}$$

and $p_{i,j-1}q_{ij} - p_{ij}q_{i,j-1} = (-1)^j$ for $i = 0, 1, 2, \ldots$ and $j = 1, 2, \ldots, a_{i+2} - 1$. If the range of j in (3) is extended to $0 \le j \le a_{i+2} - 1$, then the principal convergents are included among the intermediate convergents. We shall refer to both kinds simply as *convergents*—those in (4) as *even-indexed convergents* and those in (5) as *odd-indexed convergents*.

We shall use the notation (( )) for the fractional-part function, defined by $((x)) = x - \lfloor x \rfloor$.

Theorem 1: Suppose p/q is a convergent to a positive irrational number α. Then for $k = 1, 2, \ldots, q-1$, the sum $((k\alpha)) + (((q-k)\alpha))$ is invariant of k; in fact,

$$((k\alpha)) + (((q-k)\alpha)) = \begin{cases} ((q\alpha)) + 1 & \text{if } p/q \text{ is an even-indexed convergent,} \\ ((q\alpha)) & \text{if } p/q \text{ is an odd-indexed convergent.} \end{cases}$$

Proof: Suppose p/q is an even-indexed convergent and $1 \le k \le q-1$. Then $p/q < \alpha$, so that

$$kp/q < k\alpha. \tag{6}$$

Suppose there is an integer h such that $kp/q \le h < k\alpha$. Then

$$p/q < h/k < \alpha. \tag{7}$$

However, as an even-indexed convergent to α, the rational number p/q is the best lower approximate (as defined in [1]), which means that $k \ge q$ in (7). This contradiction to the hypothesis, together with (6), shows that

$$((kp/q)) < ((k\alpha)). \tag{8}$$

Since $1 \le q-k \le q-1$, we also have $1 = ((kp/q)) + (((q-k)p/q)) < ((k\alpha)) + (((q-k)\alpha))$. Since $((k\alpha)) + (((q-k)\alpha))$ has the same fractional part as $q\alpha$, we conclude that

$$((k\alpha)) + (((q-k)\alpha)) = ((q\alpha)) + 1.$$

The proof for odd-indexed convergents p/q is similar and omitted. □

Theorem 2: Suppose $\Delta(n) = \lfloor n\alpha \rfloor - \lfloor (n-1)\alpha \rfloor$ for some positive irrational number, for $n = 1, 2, 3, \ldots$. Then Δ is a palindromic sequence.

Proof: By Theorem 1, if p/q is an odd-indexed convergent to α, then

$$((k\alpha)) + (((q-k)\alpha)) = ((q\alpha)) \text{ for } k = 1, 2, \ldots, q-1,$$

and clearly this holds for $k = q$, also. Consequently,

$$\lfloor k\alpha \rfloor + \lfloor (q-k)\alpha \rfloor = \lfloor q\alpha \rfloor,$$
$$\sigma(k) + \sigma(q-k) = \sigma(q),$$

for $k = 1, 2, \ldots, q$. By the lemma, Δ is a palindromic sequence. □

Example 1: There is only one positive irrational number for which all the convergents are principal convergents, shown here along with its continued fraction:

$$\alpha = (1+\sqrt{5})/2 = [\![1, 1, 1, \ldots]\!].$$

The convergents are quotients of consecutive Fibonacci numbers, and the sequence σ given by $\sigma(n) = \lfloor n\alpha \rfloor$ begins with 0, 1, 3, 4, 6, 8, 9, 11, 12, 14, 16, 17, 19, 21, 22, 24, 25, 27, 29, 30, 32, 33, 35, 37, 38, 40, so that the difference sequence Δ begins with 1, 2, 1, 2, 2, 1, 2, 1, 2, 2, 1, 2, 2, 1, 2, 1, 2, 2, 1, 2, 1, 2, 2, 1, 2. The sequence Δ is palindromic, since $(\Delta(1), \ldots, \Delta(q))$ is a palindrome for

$$q \in \{1, 3, 8, 21, 55, 144, 377, 987, \ldots\}.$$

Moreover, $(\Delta(2), \ldots, \Delta(q-1))$ is a palindrome for

$$q \in \{2, 5, 13, 34, 89, 233, 610, \ldots\}.$$

In both cases, Fibonacci numbers abound.

Example 2: For $\alpha = e$, approximately 2.718281746, the continued fraction is

$$[\![2,1,2,1,1,4,1,1,6,1,1,8,1,1,10,1,1,12,1,1,\ldots]\!],$$

and the first twenty convergents (both principal and intermediate) are:

$$\begin{array}{rr}
2/1 = p_{00}/q_{00} & 106/39 = p_{60}/q_{60} \\
3/1 = p_{10}/q_{10} & 193/71 = p_{70}/q_{70} \\
5/2 = p_{01}/q_{01} & 299/110 = p_{61}/q_{61} \\
8/3 = p_{20}/q_{20} & 492/181 = p_{62}/q_{62} \\
11/4 = p_{30}/q_{30} & 685/252 = p_{63}/q_{63} \\
19/7 = p_{40}/q_{40} & 878/323 = p_{64}/q_{64} \\
30/11 = p_{31}/q_{31} & 1071/394 = p_{65}/q_{65} \\
49/18 = p_{32}/q_{32} & 1264/465 = p_{80}/q_{80} \\
68/25 = p_{33}/q_{33} & 1457/536 = p_{90}/q_{90} \\
87/32 = p_{50}/q_{50} & 2721/1001 = p_{81}/q_{81}
\end{array}$$

Here, Δ begins with 2, 3, 3, 2, 3, 3, 3, 2, 3, 3, 2, 3, 3, 3, 2, 3, 3, 2, 3, 3, 3, 2, 3, 3, 2, 3, 3, 3, 2, 3, 3, 2, and $(\Delta(1), \ldots, \Delta(q))$ is a palindrome for

$$q \in \{1, 4, 11, 18, 25, 32, 71, 536, \ldots\},$$

and $(\Delta(2), \ldots, \Delta(q-1))$ is a palindrome for

$$q \in \{2, 3, 7, 39, 110, 181, 252, 323, 394, 465, 1001, \ldots\}.$$

Opportunities: The foregoing theorems and examples suggest the problem of describing *all* the palindromes within the difference sequence Δ given by $\Delta(n) = \lfloor n\alpha \rfloor - \lfloor n\alpha - \alpha \rfloor$ for irrational α. One might then investigate what happens when $n\alpha$ is replaced by $n\alpha + \beta$, where β is a real number.

REFERENCES

1. Clark Kimberling. "Best Lower and Upper Approximates to Irrational Numbers." *Elemente der Mathematik* **52** (1997):122-26.
2. Serge Lang. *Introduction to Diophantine Approximations*. Reading, Mass.: Addison-Wesley, 1966.

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