OBTAINING DIVIDING FORMULAS $n|Q(n)$ FROM ITERATED MAPS

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1. INTRODUCTION

In this work we show that we may use iterated maps to understand and generate a dividing formula $n|Q(n)$, where n is any positive integer. A well-known example of a dividing formula concerning Fibonacci numbers, for instance, is

$$n|(F_{n+1}+F_{n-1}-1), \tag{1.1}$$

where n is a prime and F_n is the n^{th} Fibonacci number.

We will show that from iterated maps we have a systematic way to construct functions $Q(n)$ such that $n|Q(n)$. In this paper, we show how to derive the above dividing formula from an iterated map. We also generalize the result to the case in which n is any positive integer and to the case of Fibonacci numbers of degree m. We begin with Theorem 2.1 below.

2. THE FUNDAMENTAL THEOREM: $n|N(n)$

Theorem 2.1: For an iterated map,

$$n|N(n), \tag{2.1}$$

where $N(n)$ is the number of period-n points for the map.

Proof: If $N(n)=0$, formula (2.1) is obvious. If $N(n)\neq 0$, then the orbit of a period-n point is an n-cycle containing n distinct period-n points. Since there are no common elements in any two distinct n-cycles, $N(n)$ must be a multiple of n, i.e., $n|N(n)$, and $N(n)/n$ is an integer representing the number of n-cycles for the map.

As a consequence of this fundamental theorem, *each iterated map*, in principle, offers a desired $Q(n)$ function such that $n|Q(n)$, where $Q(n)=N(n)$, the number of period-n points of an iterated map. Therefore, we have an additional way to understand the dividing formula $n|Q(n)$ from the point of view of iterated maps.

3. THE $N(n)$ OF AN ITERATED MAP

For a general discussion, we consider a map $f(x)$ in some interval. The fixed points of f are determined from the formula

$$f(x)=x. \tag{3.1}$$

The number of fixed points for f can be determined from the number of intersections of the curve $y=f(x)$ with the diagonal line $y=x$ in the interval. We define $f^{[n]}(x)$ for the n^{th} iterate of x for f, then $f^{[n]}(x)\equiv f(f^{[n-1]}(x))$. We should distinguish a fixed point of $f^{[n]}$, and a period-n point of f. The fixed points of $f^{[n]}$ are determined from the formula $f^{[n]}(x)=x$; however, the period-n points of f are determined from the following two equations:

$$f^{[n]}(x)=x, \tag{3.2}$$

$$f^{[i]}(x)\neq x \quad \text{for } i=1,2,\ldots,n-1. \tag{3.3}$$

We need (3.3) because an x satisfying only (3.2) is not necessarily a period-n point of f, since it could be a fixed point of f or, in general, a period-m point of f, where $m<n$ and $m|n$. Formulas (3.2) and (3.3) together ensure that x is a period-n point of f. Then $N(n)$ represents the number of points satisfying both (3.2) and (3.3). We let $N_\Sigma(n)$ represent the number of points satisfying only (3.2); hence, $N_\Sigma(n)$ represents the number of fixed points for $f^{[n]}$. Accordingly, we have

$$N_\Sigma(n)=\sum_{d|n} N(d),$$

where the sum is over all the divisors of n (including 1 and n). $N_\Sigma(n)$ is simply determined from intersections of the curve $y=f^{[n]}(x)$ with the diagonal line. Therefore, what we obtain directly from an iterated map is not $N(n)$ but $N_\Sigma(n)$. We need a reverse formula expressing $N(n)$ in terms of $N_\Sigma(d)$. This has already been done, because we know the following two formulas from [2] and [7]:

$$N_\Sigma(n)=\sum_{d|n} N(d), \tag{3.4}$$

and

$$N(n)=\sum_{d|n}\mu(n/d)N_\Sigma(d)=\sum_{d|n}\mu(d)N_\Sigma(n/d), \tag{3.5}$$

where $\mu(d)$ is the Möbius function. $N_\Sigma(n)$ is called the *Möbius transform* of $N(n)$, and $N(n)$ is the *inverse Möbius transform* of $N_\Sigma(n)$. Hence, after calculating the $N_\Sigma(n)$ of an iterated map, we obtain a dividing formula $n|N(n)$ from (3.5). There are examples of iterated maps for which $N_\Sigma(n)$ are calculated (see [1], [5]). We summarize these in the following theorem.

Theorem 3.1: For an iterated map,

$$n|N(n), \text{ with } N(n)=\sum_{d|n}\mu(n/d)N_\Sigma(d) \tag{3.6}$$

and, especially,

$$n|(N_\Sigma(n)-N_\Sigma(1)) \quad \text{for } n \text{ a prime}. \tag{3.7}$$

4. APPLICATIONS

We consider the $B(\mu;x)$ map defined by

$$B(\mu;x)=\begin{cases}\mu x & \text{for } 0\le x\le 1/2,\\ \mu(x-1/2) & \text{for } 1/2<x\le 1,\end{cases} \tag{4.1}$$

where μ is the parameter whose value is restricted to the range $0\le\mu\le 2$ so that an x in the interval [0, 1] is mapped to the same interval. We now have the following theorem.

Theorem 4.1: Line segments in $B^{[n]}(\mu;x)$ are all parallel with slope μ^n.

Proof: In the beginning, for a given μ, $B(\mu;x)$ contains two parallel line segments with slope μ. From (4.1), we see that each line segment will multiply its slope by a factor μ after one iteration. Q.E.D.

We consider the following cases.

4.1 The Case in Which $x = 1/2$ Is a Period-2 Point

If $x = 1/2$ is a period-2 point of the $B(\mu)$ map, it requires that $B^2(\mu, 1/2) = 1/2$. This then requires that $\mu > 1$ and $\mu^2 - \mu - 1 = 0$. Solving this, we have $\mu = (1+\sqrt{5})/2 \approx 1.618$, the well-known golden mean. We denote this μ by Σ_2, indicating that for this parameter value $x = 1/2$ is a period-2 point. To obtain $N_\Sigma(n)$, we need to count the number of line segments in $B^{[n]}(\mu)$ that intersect the diagonal line. Detailed discussions of this map can be seen in [3] and [4]. Briefly, we see that starting from $x_0 = 1/2$, we have a 2-cycle, $\{x_0, x_1\}$, where $x_1 = \mu/2$. It follows that there are two types of line segments. We denote by L the type of line segments connecting points $(x_a, 0)$ and $(x_b, \mu/2)$ with $0 \le x_a < x_b \le 1$, and denote by S the type of line segments connecting points $(x_c, 0)$ and $(x_d, 1/2)$ with $0 \le x_c < x_d \le 1$. Since $x_0 \to x_1$ and $x_1 \to x_0$ under an iteration, it follows that the behavior of line segments of these two types under iteration is

$$L \to L + S \quad \text{and} \quad S \to L. \tag{4.2}$$

Using the symbols L and S, we see that the graph of $B(\mu)$ contains two L. (4.2) shows how the number of line segments increases under the action of iteration. Let $L(n)$ and $S(n)$ be the number of line segments of type L and S in $B^{[n]}(\mu)$, respectively. (4.2) shows simply that each L is from previous L and S, so $L(n) = L(n-1) + S(n-1)$, and each S is from previous L, so $S(n) = L(n-1)$. From these, we conclude that $L(n) = L(n-1) + L(n-2)$ and $S(n) = S(n-1) + S(n-2)$. That is, $L(n)$ and $S(n)$ are both the type of sequences of which each element is the sum of its previous two elements. Starting with an L, according to (4.2), the orbit of which is

$$L \to LS \to 2LS \to 3L2S \to 5L3S \to 8L5S \to \cdots,$$

we easily see that $L(n) = F_n$ and $S(n) = F_{n-1}$. In conclusion, starting from an L, under the action of iteration there are F_n (L-type) and $S(n)$ (S-type) parallel line segments generated in $B^{[n]}(\mu)$. We will use this result. $N_\Sigma(n)$ is then determined from the number of intersections of these line segments with the diagonal line.

Consider first the S-type line segments. We note that all S-type line segments in $B^{[n]}(\mu)$ are parallel and can be divided into two parts, one part in the range $0 \le x \le 1/2$ and the other in the range $1/2 < x \le 1$. We easily see that each line segment in the range $0 \le x \le 1/2$ intersects the diagonal line once, and others in the range $1/2 < x \le 1$ cannot intersect the diagonal line. The original line segment in the range $0 \le x \le 1/2$ is an L, so the number of S-type line segments in $B^{[n]}(\mu)$ in this range is $S(n) = F_{n-1}$.

Consider next the L-type line segment. Similarly, line segments of this type in $B^{[n]}(\mu)$ are parallel, and each of those that is in the range $0 \le x \le \mu/2$ intersects the diagonal line once. Others in the range $\mu/2 \le x \le 1$ cannot intersect the diagonal line. We divide the range $0 \le x \le \mu/2$ into $0 \le x \le 1/2$ and $1/2 < x \le \mu/2$. The original line segment in the range $0 \le x \le 1/2$ is an L, so the number of L-type line segments in $B^{[n]}(\mu)$ in this range is $L(n) = F_n$. Next, the original line segment in the range $1/2 < x \le \mu/2$ is an S. After one iteration, $S \to L$; the L in the right-hand side, after $n-1$ iterations, generates all the L-type line segments in $B^{[n]}(\mu)$ in this range, the number of which is therefore $L(n-1) = F_{n-1}$.

In all, the total number of intersections of line segments of types L and S with the diagonal line is thus $F_{n-1} + F_n + F_{n-1} = F_{n+1} + F_{n-1}$. We conclude that

$$N_\Sigma(n) = F_{n+1} + F_{n-1} = F_n + 2F_{n-1}. \tag{4.3}$$

$N_\Sigma(n)$ is, in fact, the Lucas number L_n. From (3.6) and (3.7), we have

$$n|N(n), \quad \text{with } N(n) = \sum_{d|n} \mu(n/d)(F_{n+1} + F_{n-1}) \tag{4.4}$$

and

$$n|(F_{n+1} + F_{n-1} - 1), \quad \text{for } n \text{ a prime.} \tag{4.5}$$

Formula (4.5) is also a known result [6]; however, we see that it is traceable from the point of view of iterated maps.

For n a composite number, (4.4) offers additional relations for Fibonacci numbers or Lucas numbers. This result seems to be a new one. Consider, as a simple example, taking $n = 12$; since $N(12) = 300$, we can easily check that $12|300$.

4.2 The Case in Which $x = 1/2$ Is a Period-m Point

We now consider the general case in which $x = 1/2$ is a period-m point. In general, there are many values of μ in the range $0 \le \mu \le 2$ such that $x = 1/2$ is a period-m point. If there are k such parameter values, we denote these by $\mu_1 < \mu_2 < \mu_3 < \cdots < \mu_k \equiv \Sigma_m$. We will choose the largest one as the parameter value, i.e., $\mu = \Sigma_m$. It follows that Σ_m satisfies the equation:

$$\mu^m - \sum_{i=0}^{m-1} \mu^i = 0. \tag{4.6}$$

For instance, we have $\Sigma_3 \cong 1.839$, $\Sigma_4 \cong 1.9276$. We refer the reader to [4] for details. Briefly, we see that, starting from $x_0 = 1/2$, we have an m-cycle, $\{x_0, x_1, \ldots, x_{m-1}\}$, where $x_i = f^{[i]}(x_0)$ is the i^{th} iterate of x_0. It is convenient also to define $x_m = x_0$. We have, for instance, $x_1 = (1/2)\mu$, and from (4.6) we have $x_1 = (1/2)(1 + 1/\mu + 1/\mu^2 + \cdots + 1/\mu^{m-1})$. We list all the values of x_i in this way:

$$\begin{aligned}
x_1 &= (1/2)(1 + 1/\mu + 1/\mu^2 + \cdots + 1/\mu^{m-1}), \\
x_2 &= (1/2)(1 + 1/\mu + 1/\mu^2 + \cdots + 1/\mu^{m-2}), \\
&\ldots, \\
x_{m-2} &= (1/2)(1 + 1/\mu + 1/\mu^2), \\
x_{m-1} &= (1/2)(1 + 1/\mu), \\
x_m &= x_0 = 1/2.
\end{aligned} \tag{4.7}$$

We see that $x_1 > x_2 > x_3 > \cdots > x_{m-1} > x_m$. It follows that there are m types of line segments. We denote by L_i the type of line segments connecting points $(x_a, 0)$ and (x_b, x_i), where $0 \le x_a < x_b \le 1$ and $1 \le i \le m$. The behavior of line segments of these types after one iteration is

$$\begin{aligned}
&L_1 \to L_1 + L_2, \\
&L_2 \to L_1 + L_3, \\
&\ldots, \\
&L_{m-1} \to L_1 + L_m,
\end{aligned} \tag{4.8}$$

and

$$L_m \to L_1. \tag{4.9}$$

We note that L_m (or L_0) is the only type of line segment that does not break into two line segments after one iteration. The original graph of $B(\mu)$ contains two L_1. Equations (4.8) and (4.9) show how the number of line segments increases under the action of iteration. Let $L_i(n)$ be the number of line segments of type L_i in $B^{[n]}(\mu)$, then (4.8) and (4.9) show that

$$\begin{aligned} &L_2(n) = L_1(n-1), \\ &L_3(n) = L_2(n-1) = L_1(n-2), \\ &L_4(n) = L_3(n-1) = L_1(n-3), \\ &\quad\ldots, \\ &L_m(n) = L_{m-1}(n-1) = L_1(n-m+1), \end{aligned}$$

or

$$L_j(n) = L_{j-1}(n-1) = L_1(n-j+1), \quad \text{for } 2 \le j \le m, \tag{4.10}$$

and especially,

$$L_1(n) = \sum_{j=1}^{m} L_j(n-1) = \sum_{j=1}^{m} L_1(n-j). \tag{4.11}$$

It follows that all these $L_i(n)$ are sequences of the following type:

$$L_i(n) = \sum_{j=1}^{m} L_i(n-j), \quad 1 \le i \le m, \tag{4.12}$$

i.e., each element of which is the sum of its previous m elements. Starting with an L_1, according to (4.8) and (4.9), the orbit of L_1 is $L_1 \to L_1L_2 \to 2L_1L_2L_3 \to \cdots$. We see that $L_1(n) = F_n^{(m)}$, the Fibonacci numbers of degree m, whose definition is

$$F_n^{(m)} = \sum_{i=1}^{m} F_{n-i}^{(m)},$$

with the first m elements defined by

$$F_1^{(m)} = 1 \quad \text{and} \quad F_i^{(m)} = 2^{i-2} \quad \text{for } 2 \le i \le m. \tag{4.13}$$

Conventionally, we define $F_j^{(m)} = 0$ for $j \le 0$.

We conclude that, starting from an L_1, we have $L_1(n) = F_n^{(m)}$ and, in general, $L_i(n) = F_{n-i+1}^{(m)}$, where $1 \le i \le m$. In order to discuss the number of intersections of these line segments with the diagonal line, we need to know, starting from an L_a-type line segment, how many L_b-type line segments there are in $B^{[n]}(\mu)$. We let $L_b(n, L_a)$ represent the number of L_b-type line segments generated after n iterations of a starting line segment L_a. Using this notation, we have

$$L_b(n, L_1) = F_{n-b+1}^{(m)}, \tag{4.14}$$

and we have the following theorem.

Theorem 4.2: $L_b(n, L_a) = \sum_{s=0}^{m-a} F_{n-b-s}^{(m)}$, where $1 \le a,\ b \le m$.

Proof: We will prove this theorem by induction. We start from a line segment L_a and calculate the number of L_b-type line segments generated after n iterations of L_a. Consider first $L_a = L_m$. To calculate $L_b(n, L_m)$, we note that after one iteration we have $L_m \to L_1$. The L_1 in the right-hand side, after $n-1$ iterations, generates all the L_b-type line segments in $B^{[n]}(\mu)$ in this range, the number of which is $L_b(n-1, L_1)$. Using (4.14), we conclude that

$$L_b(n, L_m) = L_b(n-1, L_1) = F_{n-b}^{(m)} = \sum_{s=0}^{m-a} F_{n-b-s}^{(m)}, \quad \text{where } a = m. \tag{4.15}$$

Consider next $L_a = L_{m-1}$. After one iteration, $L_{m-1} \to L_1 + L_m$. The L_1 and L_m in the right-hand side, after $n-1$ iterations, generate all the L_b-type line segments in $B^{[n]}(\mu)$ in this range, the number of which is $L_b(n-1, L_1) + L_b(n-1, L_m)$. Using (4.14) and (4.15), we conclude that

$$\begin{aligned} L_b(n, L_a) &= L_b(n-1, L_1) + L_b(n-1, L_m) \\ &= F_{n-b}^{(m)} + F_{n-b-1}^{(m)} = \sum_{s=0}^{m-a} F_{n-b-s}^{(m)}, \quad \text{where } a = m-1. \end{aligned} \tag{4.16}$$

We can now prove the general case by induction. Consider $L_a = L_{m-i}$, and suppose that

$$L_b(n, L_a) = \sum_{s=0}^{m-a} F_{n-b-s}^{(m)}, \quad \text{where } a = m-i. \tag{4.17}$$

Now consider $L_a = L_{m-i-1}$. After one iteration, $L_{m-i-1} \to L_1 + L_{m-i}$. The L_1 and L_{m-i} in the right-hand side, after $n-1$ iterations, generates all the L_b-type line segments in $B^{[n]}(\mu)$ in this range, the number of which is $L_b(n-1, L_1) + L_b(n-1, L_{m-i})$. Using (4.14) and (4.16), we conclude that

$$\begin{aligned} L_b(n, L_a) &= L_b(n-1, L_1) + L_b(n-1, L_{m-i}) \\ &= F_{n-b}^{(m)} + \sum_{s=0}^{i} F_{n-b-s-1}^{(m)} = \sum_{s=0}^{m-a} F_{n-b-s}^{(m)}, \quad \text{where } a = m-i-1. \text{ Q.E.D.} \end{aligned} \tag{4.18}$$

With Theorem 4.2 established, we can now discuss the intersections of line segments of these m-types in $B^{[n]}(\mu)$ with the diagonal line.

We consider line segments of type L_b in $B^{[n]}(\mu)$. All these line segments are parallel and can be divided into two parts; one part is in the range $0 \le x \le x_b$, the other is in the range $x_b < x \le 1$. We easily see that each line segment of those in the range $0 \le x \le x_b$ intersects the diagonal line once and others in the range $x_b < x \le 1$ cannot intersect the diagonal line. We divide the range $0 \le x \le x_b$ into $0 \le x \le 1/2$ and $1/2 < x \le x_b$.

Consider first the range $0 \le x \le 1/2$. The original line segment in this range is an L_1. The number of line segments of type L_b in $B^{[n]}(\mu; x)$ in this range is $L_b(n, L_1) = F_{n-b+1}^{(m)}$.

Consider next the range $1/2 < x \le x_b$. The original line segment in the range $1/2 < x \le x_b$ is L_{b+1}. The number of L_b-type line segments in $B^{[n]}(\mu)$ in this range is $L_b(n, L_{b+1}) = \sum_{s=0}^{m-b-1} F_{n-b-s}^{(m)}$.

Therefore, the total number of intersections of these m-type line segments with the diagonal line is

$$\begin{aligned} \sum_{b=1}^{m} \left[F_{n-b+1}^{(m)} + \sum_{s=0}^{m-b-1} F_{n-b-s}^{(m)} \right] &= \sum_{b=1}^{m} F_{n-b+1}^{(m)} + \sum_{b=1}^{m} \sum_{s=0}^{m-b-1} F_{n-b-s}^{(m)} \\ &= \sum_{b=0}^{m-1} F_{n-b}^{(m)} + \sum_{k=1}^{m-1} k F_{n-k}^{(m)} = \sum_{k=0}^{m-1} (k+1) F_{n-k}^{(m)} \end{aligned} \tag{4.19}$$

$$\equiv L_n^{(m)}. \tag{4.20}$$

$L_n^{(m)}$, defined above, may be called the Lucas numbers of degree m, whose definition is

$$L_n^{(m)} = \sum_{j=1}^{m} L_{n-j}^{(m)}, \text{ with the first } m \text{ elements defined by}$$

$$L_i^{(m)} = 2^i - 1, \quad 1 \le i \le m. \tag{4.21}$$

We then have the final results:

$$N_\Sigma(n) = L_n(m), \tag{4.22}$$

$$n|N(n), \text{ with } N(n) = \sum_{d|n} \mu(n/d) L_n^{(m)}, \tag{4.23}$$

and

$$n|(L_n^{(m)} - 1), \text{ for } n \text{ a prime.} \tag{4.24}$$

5. CONCLUSIONS

Many $N(n)$ such that $n|N(n)$ can be obtained in this way for other iterated maps. In principle, infinite $N(n)$ can be obtained, since each iterated map contributes an $N(n)$. It seems that the existence of dividing formulas is not so rare and not so mysterious.

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