THE FIBONACCI TRIANGLE MODULO p

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(Submitted July 1996-Final Revision August 1997)

1. INTRODUCTION

Let $F_n! = F_nF_{n-1}\dots F_2F_1$.

Definition: The Fibonacci coefficient $\left[{n \atop k}\right]_{\mathfrak{F}}$ is defined to be

$$\begin{bmatrix} n \\ k \end{bmatrix}_{\mathfrak{F}} = \frac{F_n!}{F_k!F_{n-k}!} = \frac{F_nF_{n-1}\dots F_1}{(F_kF_{k-1}\dots F_1)(F_{n-k}F_{n-k-1}\dots F_1)}.$$

An important property of the Fibonacci coefficients from [4] is

$$\begin{bmatrix} k \\ \ell \end{bmatrix}_{\mathfrak{F}} = F_{\ell+1}\begin{bmatrix} k-1 \\ \ell \end{bmatrix}_{\mathfrak{F}} + F_{k-\ell-1}\begin{bmatrix} k-1 \\ \ell-1 \end{bmatrix}_{\mathfrak{F}}. \tag{1}$$

From the Fibonacci coefficients we form the Fibonacci triangle in much the same way as Pascal's triangle is formed from the binomial coefficients; namely, the Fibonacci triangle is formed by letting the k^{th} element of the n^{th} row be $\left[{n \atop k}\right]_{\mathfrak{F}}$.

$$\begin{array}{ccccccccccc}
&&&&&1&&&&& \\
&&&&1&&1&&&& \\
&&&1&&1&&1&&& \\
&&1&&2&&2&&1&& \\
&1&&3&&6&&3&&1& \\
1&&5&&15&&15&&5&&1
\end{array}$$

FIGURE 1. Rows 0 to 5 of the Fibonacci Triangle

The parity of the binomial coefficients and the iterative structure of Pascal's triangle have been the subject of many papers (see, e.g., [2], [3], [13]). More recently, the Fibonacci coefficients and the iterative structure of the Fibonacci triangle modulo 2 and 3 has been examined in [5], [11], and [12]. In this paper we extend the results of [11] and [12] from the Fibonacci coefficients and triangle modulo 2 and 3 to modulus p for p an odd prime.

For an odd prime p other than 5 and $i \geq 0$, define $r_i \in \mathbb{N}$ as the smallest number such that $p^i | F_{r_i}$. In particular, $r_0 = 1$ and r_1 is what is commonly called the rank of apparition of p. We will denote $\wp = \{r_0, r_1, \dots\}$. It is well known that $r_i | r_{i+1}$ for all $i \in \mathbb{N}$, so any $n \in \mathbb{N}$ can be written uniquely as $n = n_kr_k + n_{k-1}r_{k-1} + \dots + n_1r_1 + n_0$ for $0 \leq n_i < \frac{r_{i+1}}{r_i}$. We call this the base $\wp$ representation of $n \in \mathbb{N}$.

Our main results are

Theorem 1: Let $r = \max_{i\geq 0}\frac{r_{i+1}}{r_i}$. The number of entries in the n^{th} row of the Fibonacci triangle not divisible by p is $2^{s_1}3^{s_2}4^{s_3}\dots r^{s_{r-1}}$, where s_i is the number of i's in the base $\wp$ expansion of n.

Theorem 2: Let $p \neq 2, 5$ be a prime. There is the following connection between the Fibonacci and binomial coefficients modulo p:

$$\begin{bmatrix} nr_1 \\ kr_1 \end{bmatrix}_{\mathfrak{F}} \equiv \binom{n}{k} F_{r_1+1}^{k(n-k)r_1} \pmod{p}.$$

In particular, the triangle Δ_{F_p} formed by having $\left[\begin{smallmatrix} nr_1 \\ kr_1 \end{smallmatrix}\right]_{\mathfrak{F}} \pmod{p}$ as the k^{th} entry of the n^{th} row is Pascal's triangle modulo p if and only if r_1 is even.

Theorem 3: For $p \neq 2, 5$ a prime, we have

$$\begin{bmatrix} nr_1 + j \\ mr_1 + i \end{bmatrix}_{\mathfrak{F}} \equiv \binom{n}{m} \begin{bmatrix} j \\ i \end{bmatrix}_{\mathfrak{F}} F_{r_1+1}^{r_1 m(n-m)+i(n-m)+m(j-i)} \pmod{p}.$$

2. PRELIMINARY FACTS

Of fundamental importance in our investigation are the following two well-known facts (see [9]): First, if (a, b) denotes the greatest common divisor of two natural numbers, then

$$(F_n, F_m) = F_{(m,n)}. \tag{2}$$

Second,

$$F_{n+m} = F_n F_{m-1} + F_{n+1} F_m. \tag{3}$$

A sequence $\{A_j\}$ is said to be regularly divisible by $d \in \mathbb{N}$ if there exists $r(d) \in \mathbb{N}$ such that $d \mid A_j$ if and only if $r(d) \mid j$. A sequence is regularly divisible if it is regularly divisible for all $d \in \mathbb{N}$ (see [5]). From (2), we see that the sequence $\{F_n\}_{n=1}^{\infty}$ is regularly divisible. To simplify notation, for p our fixed prime and for $i \geq 0$, we let $r_i \in \mathbb{N}$ be the smallest number such that $p^i \mid F_{r_i}$. Notice that $r_0 = 1$ and r_1 is what is generally called the rank of apparition of p. Let $\wp = \{r_0, r_1 \ldots\}$. Since the Fibonacci sequence is regularly divisible $r_i \mid r_{i+1}$ so each $n \in \mathbb{N}$ can be written uniquely as $n = n_t r_t + n_{t-1} r_{t-1} + \cdots + n_1 r_1 + n_0$ with $0 \leq n_i < \frac{r_{i+1}}{r_i}$. We call this the base $\wp$ representation of n and denote it by $n = (n_t n_{t-1} \ldots n_1 n_0)_{\wp}$ (see [6]).

It is well known from [7] that for $i \geq 1$ we have

$$\frac{r_{i+1}}{r_i} = \begin{cases} 1 \\ \text{or} \\ p. \end{cases} \tag{4}$$

The following theorem was first shown in [5] in a different form. The introduction of the base $\wp$ allows us to state the theorem more succinctly. The theorem was given in this form in [10]. The proof is reproduced here with the permission of the first author of [10].

Kummer's Theorem for Generalized Binomial Coefficients: Let $\mathcal{A} = \{A_j\}_{j=1}^{\infty}$ be a sequence of positive integers. If $\mathcal{A}$ is regularly divisible by the powers of p, then the highest power of p that divides

$$\begin{bmatrix} m+n \\ m \end{bmatrix}_{\mathcal{A}} = \frac{A_{m+n} A_{m+n-1} \cdots A_{n+1}}{A_m A_{m-1} \cdots A_2 A_1}$$

is the number of carries that occur when the integers n and m are added in base $\wp$, where $\wp = \{r_j\}_{j=0}^{\infty}$ for r_j defined by $p^j \mid A_{r_j}$, $p^j \nmid A_r$ for $0 < r < r_j$.

Proof: By definition of r_i, A_{r_i} is the first element in $\mathcal{A}$ divisible by p^i. By regular divisibility of the sequence $\{A_j\}_{j=1}^{\infty}$, we see that $p^i \mid A_k$ if and only if $r_i \mid k$. This means the number of A_k, $k \leq n$ that are multiples of p^i is

$$\left\lfloor \frac{n}{r_i} \right\rfloor = \left\lfloor \frac{n_t r_t + \cdots + n_1 r_1 + n_0}{r_i} \right\rfloor = n_t \frac{r_t}{r_i} + \cdots + n_{i+1} \frac{r_{i+1}}{r_i} + n_i.$$

Now suppose, in base $\wp$, we have $m = m_t r_t + m_{t-1} r_{t-1} + \cdots + m_1 r_1 + m_0$ and $n = n_t r_t + n_{t-1} r_{t-1} + \cdots + n_1 r_1 + n_0$, where we allow some of the initial digits to be 0 so we may assume m and n are written with the same number of digits in base $\wp$. Counting the multiples of p^i in $\{A_1, A_2, \ldots, A_{m+n}\}$, $\{A_1, A_2, \ldots, A_m\}$, and $\{A_1, A_2, \ldots, A_n\}$, we see a carry at the i^{th} place,

$$(m_{i-1} r_{i-1} + \cdots + m_1 r_1 + m_0) + (n_{i-1} r_{i-1} + \cdots + n_1 r_1 + n_0) \geq r_i$$

occurs if and only if the number of multiples of p^i in $\{A_1, A_2, \ldots, A_{m+n}\}$ is one greater than the number of multiples of p^i in $\{A_1, A_2, \ldots, A_m\}$ plus the number of multiples of p^i in $\{A_1, A_2, \ldots, A_n\}$. Therefore, the number of carries is the highest power of p that divides $\left[{m+n \atop m}\right]_{\mathcal{A}}$. □

In particular, the theorem applies to the Fibonacci sequence: $\{\mathcal{A}_j\}_{j=1}^{\infty} = \{F_j\}_{j=1}^{\infty}$.

Corollary (Knuth and Wilf) [5]: The highest power of p that divides $\left[{m+n \atop n}\right]_{\mathfrak{F}}$ is the number of carries that occur when the integers n and m are added in base $\wp$, where $\wp = \{r_j\}_{j=0}^{\infty}$ for r_j defined by $p^j \mid F_{r_j}$, $p^j \nmid F_r$ for $0 < r < r_j$.

3. CONGRUENCES FOR FIBONACCI NUMBERS AND COEFFICIENTS

In this section we give a series of lemmas about congruences of Fibonacci numbers and coefficients.

Lemma 1: For $i \geq 1$, $F_{nr_i+1} \equiv F_{nr_i-1} \equiv F_{r_i+1}^n \pmod{p^i}$.

Proof: Since $p^i | F_{nr_i}$, we have $F_{nr_i+1} = F_{nr_i} + F_{nr_i-1} \equiv F_{nr_i-1} \pmod{p^i}$, so we will switch freely between F_{nr_i+1} and F_{nr_i-1} modulo p^i throughout the rest of the article. Since $p^i \mid F_{r_i}$, $F_{r_i+1}^1 = F_{r_i} + F_{r_i-1} \equiv F_{1r_i-1} \pmod{p^i}$, so the lemma is true for $n = 1$. Assume $F_{kr_i-1} \equiv F_{r_i+1}^k \pmod{p^i}$. Using (3) with $n = kr_i$, $m = r_i + 1$ gives

$$F_{(k+1)r_i-1} \equiv F_{(k+1)r_i+1} \equiv F_{kr_i} F_{r_i} + F_{kr_i+1} F_{r_i+1} \equiv F_{kr_i+1} F_{r_i+1} \equiv F_{r_i+1}^{k+1} \pmod{p^i}. \quad \square$$

Lemma 2: For $i \geq 1$, $F_{nr_i} \equiv F_{r_i}(nF_{r_i+1}^{n-1}) \pmod{p^{2i}}$.

Proof: This is clearly true for $n = 1$. Now assume $F_{kr_i} \equiv F_{r_i}(kF_{r_i+1}^{k-1}) \pmod{p^{2i}}$. Then, using (3) with $n = kr_i - 1$, $m = r_i + 1$ gives

$$F_{(k+1)r_i} = F_{kr_i-1} F_{r_i} + F_{kr_i} F_{r_i+1} \equiv F_{r_i}(F_{kr_i-1} + kF_{r_i+1}^{k-1} F_{r_i+1}) \pmod{p^{2i}}. \tag{5}$$

Since Lemma 1 says $F_{kr_i-1} \equiv F_{r_i+1}^k \pmod{p^i}$, we get

$$F_{kr_i-1} + kF_{r_i+1}^{k-1} F_{r_i+1} \equiv F_{r_i+1}^k + kF_{r_i+1}^k \equiv (k+1)F_{r_i+1}^k \pmod{p^i}.$$

Since $p^i \mid F_{r_i}$, this congruence gives

$$F_{r_i}(F_{kr_i-1} + kF_{r_i+1}^{k-1} F_{r_i+1}) \equiv F_{r_i}(k+1)F_{r_i+1}^k \pmod{p^{2i}}.$$

This congruence together with (5) gives $F_{(k+1)r_i} \equiv F_{r_i}(k+1)F_{r_i+1}^k \pmod{p^{2i}}$. □

Lemma 3: For $0 \le j, \ell$ and $0 \le m \le r_1 - 1$, we have $F_{\ell r_1+m}F_{jr_1+1} \equiv F_{\ell r_1+1}F_{jr_1+m} \pmod{p}$.

Proof: For $m=0$, both sides are congruent to 0 modulo p since $p|F_{\ell r_1}$ and $p|F_{jr_1}$. For $m=1$, both sides are identical. Assume that $F_{\ell r_1+m}F_{jr_1+1} \equiv F_{\ell r_1+1}F_{jr_1+m} \pmod{p}$ for all $m < k \le r_1 - 1$ for some k. Using our induction hypothesis, $F_{jr_1+k} = F_{jr_1+(k-1)} + F_{jr_1+(k-2)}$, and $F_{\ell r_1+k} = F_{\ell r_1+(k-1)} + F_{\ell r_1+(k-2)}$, we get

$$\begin{aligned} F_{\ell r_1+k}F_{jr_1+1} &= F_{\ell r_1+(k-1)}F_{jr_1+1} + F_{\ell r_1+(k-2)}F_{jr_1+1} \\ &\equiv F_{\ell r_1+1}F_{jr_1+(k-1)} + F_{\ell r_1+1}F_{jr_1+(k-2)} = F_{\ell r_1+1}F_{jr_1+k} \pmod{p}. \quad \square \end{aligned}$$

Note that alternate forms of Lemma 3 are

$$\frac{F_{\ell r_1+m}}{F_{\ell r_1+1}} \equiv \frac{F_{jr_1+m}}{F_{jr_1+1}} \pmod{p},$$

which will be used below in Lemma 6 and, for $m \ne 0$,

$$\frac{F_{\ell r_1+m}}{F_{jr_1+m}} \equiv \frac{F_{\ell r_1+1}}{F_{jr_1+1}} \pmod{p},$$

which we will use in Theorem 2 below.

Lemma 4: For $0 \le j, \ell$ we have

$$\frac{F_{\ell r_1+1}}{F_1} \equiv \frac{F_{(\ell+j)r_1+1}}{F_{jr_1+1}} \pmod{p}.$$

Proof: By (3) with $n = \ell r_1$, $m = jr_1 + 1$, we have

$$F_{(\ell+j)r_1+1} = F_{\ell r_1}F_{jr_1} + F_{\ell r_1+1}F_{jr_1+1} \equiv F_{\ell r_1+1}F_{jr_1+1} \pmod{p}.$$

Since F_{jr_1+1} is invertible modulo p, we may divide to put this in the form of the statement of the lemma. $\square$

Lemma 5: For $p \ne 2, 5$,

$$F_{r_1-1} \equiv \begin{cases} 1 \pmod{p} & \text{if } r_1 \equiv 2 \pmod{4}, \\ -1 \pmod{p} & \text{if } r_1 \equiv 0 \pmod{4}, \\ \text{an element of order 4} \pmod{p} & \text{if } r_1 \text{ is odd}. \end{cases}$$

Proof: From (3) with $n = a-1$, $m = a$, we get $F_a^2 + F_{a-1}^2 = F_{2a-1}$. From (3) with $n = a$, $m = a+1$, we get $F_a^2 + F_{a+1}^2 = F_{2a+1}$. If $r_1 = 2a$, then $F_{2a+1} = F_{2a} + F_{2a-1} \equiv F_{2a-1} \pmod{p}$, so

$$F_a^2 + F_{a-1}^2 = F_{2a-1} \equiv F_{2a+1} = F_a^2 + F_{a+1}^2 = 2F_a^2 + F_{a-1}^2 + 2F_aF_{a-1} \pmod{p},$$

where the last equality is found by expanding $F_{a+1}^2 = (F_a + F_{a-1})^2$. This means $0 \equiv F_a^2 + 2F_aF_{a-1} \pmod{p}$. Since $F_a \not\equiv 0 \pmod{p}$, we can factor it out to get $0 \equiv F_a + 2F_{a-1} \pmod{p}$ or, stated differently, $F_a \equiv -2F_{a-1} \pmod{p}$. Then $F_{a+1} = F_a + F_{a-1} \equiv -F_{a-1} \pmod{p}$. If $F_{a+k} \equiv (-1)^k F_{a-k} \pmod{p}$ for all $0 \le k < \ell \le a-1$, then

$$\begin{aligned} F_{a+\ell} &= F_{a+\ell-1} + F_{a+\ell-2} \equiv (-1)^{\ell-1} F_{a-\ell+1} + (-1)^{\ell-2} F_{a-\ell+2} \\ &= (-1)^{\ell}(F_{a-(\ell-2)} - F_{a-(\ell-1)}) = (-1)^{\ell} F_{a-\ell} \pmod{p}, \end{aligned}$$

so

$$F_{r_1-1} = F_{a+(a-1)} \equiv (-1)^{a-1} F_{a-(a-1)} = (-1)^{a-1} \pmod{p}.$$

This means that if $r_1 \equiv 2 \pmod 4$, we have a odd, so $F_{r_1-1} \equiv 1 \pmod{p}$. If $r_1 \equiv 0 \pmod 4$, we have a even, so $F_{r_1-1} \equiv -1 \pmod{p}$.

Now assume that r_1 is odd. Since $F_{r_1} \equiv 0 \pmod{p}$, we get $F_{r_1-1} \equiv F_{r_1+1} \pmod{p}$. Assume $F_{r_1+k} \equiv (-1)^{k-1} F_{r_1-k} \pmod{p}$ for all $0 \le k < \ell \le r_1 - 1$. Then

$$\begin{aligned} F_{r_1+\ell} = F_{r_1+(\ell-1)} + F_{r_1+(\ell-2)} &\equiv (-1)^{\ell-1}(F_{r_1-(\ell-2)} - F_{r_1-(\ell-1)}) \\ &\equiv (-1)^{\ell-1} F_{r_1-\ell} \pmod{p}. \end{aligned}$$

Therefore, $F_{2r_1-1} = F_{r_1+(r_1-1)} \equiv (-1)^{r_1-2} F_{r_1-(r_1-1)} \equiv -1 \pmod{p}$. By (3) with $n = r_1 - 1$, $m = r_1$, we get

$$F_{2r_1-1} = F_{r_1}^2 + F_{r_1-1}^2 \equiv F_{r_1-1}^2 \pmod{p},$$

so $F_{r_1-1}^2 \equiv -1 \pmod{p}$, i.e., has order 4 modulo p. □

Lemma 6: For $0 \le i \le j < r_1$,

$$\begin{bmatrix} nr_1 + j \\ mr_1 + i \end{bmatrix}_{\mathfrak{F}} \equiv \begin{bmatrix} nr_1 \\ mr_1 \end{bmatrix}_{\mathfrak{F}} \begin{bmatrix} j \\ i \end{bmatrix}_{\mathfrak{F}} F_{(n-m)r_1-1}^{i} F_{mr_1-1}^{j-i} \pmod{p}.$$

Proof: This is clear for $i = 0 = j$. Assume true for all $0 \le i \le j < k < r_1$ for some k. Take $1 \le \ell \le k - 1$. Then, by (1),

$$\begin{bmatrix} nr_1 + k \\ mr_1 + \ell \end{bmatrix}_{\mathfrak{F}} = F_{mr_1+\ell+1} \begin{bmatrix} nr_1 + k - 1 \\ mr_1 + \ell \end{bmatrix}_{\mathfrak{F}} + F_{(n-m)r_1+k-\ell-1} \begin{bmatrix} nr_1 + k - 1 \\ mr_1 + \ell - 1 \end{bmatrix}_{\mathfrak{F}}.$$

The induction hypothesis gives

$$\begin{aligned} \begin{bmatrix} nr_1 + k \\ mr_1 + \ell \end{bmatrix}_{\mathfrak{F}} &\equiv F_{mr_1+\ell+1} \begin{bmatrix} nr_1 \\ mr_1 \end{bmatrix}_{\mathfrak{F}} \begin{bmatrix} k-1 \\ \ell \end{bmatrix}_{\mathfrak{F}} F_{(n-m)r_1-1}^{\ell} F_{mr_1-1}^{k-\ell-1} + F_{(n-m)r_1+k-\ell-1} \begin{bmatrix} nr_1 \\ mr_1 \end{bmatrix}_{\mathfrak{F}} \begin{bmatrix} k-1 \\ \ell-1 \end{bmatrix}_{\mathfrak{F}} F_{(n-m)r_1-1}^{\ell-1} F_{mr_1-1}^{k-\ell} \\ &\equiv \begin{bmatrix} nr_1 \\ mr_1 \end{bmatrix}_{\mathfrak{F}} F_{(n-m)r_1-1}^{\ell} F_{mr_1-1}^{k-\ell} \left(\frac{F_{mr_1+\ell+1}}{F_{mr_1-1}} \begin{bmatrix} k-1 \\ \ell \end{bmatrix}_{\mathfrak{F}} + \frac{F_{(n-m)r_1+k-\ell-1}}{F_{(n-m)r_1-1}} \begin{bmatrix} k-1 \\ \ell-1 \end{bmatrix}_{\mathfrak{F}} \right) \pmod{p}. \end{aligned}$$

Using Lemma 3, we find this is equivalent to

$$\begin{bmatrix} nr_1 + k \\ mr_1 + \ell \end{bmatrix}_{\mathfrak{F}} \equiv \begin{bmatrix} nr_1 \\ mr_1 \end{bmatrix}_{\mathfrak{F}} F_{(n-m)r_1-1}^{\ell} F_{mr_1-1}^{k-\ell} \left(\frac{F_{\ell+1}}{F_1} \begin{bmatrix} k-1 \\ \ell \end{bmatrix}_{\mathfrak{F}} + \frac{F_{k-\ell-1}}{F_1} \begin{bmatrix} k-1 \\ \ell-1 \end{bmatrix}_{\mathfrak{F}} \right) \pmod{p}.$$

By (1), we conclude that

$$\begin{bmatrix} nr_1 + k \\ mr_1 + \ell \end{bmatrix}_{\mathfrak{F}} \equiv \begin{bmatrix} nr_1 \\ mr_1 \end{bmatrix}_{\mathfrak{F}} \begin{bmatrix} k \\ \ell \end{bmatrix}_{\mathfrak{F}} F_{(n-m)r_1-1}^{\ell} F_{mr_1-1}^{k-\ell} \pmod{p}.$$

The cases $\ell = 0$ and $\ell = k$ are dealt with similarly. □

4. MAIN RESULTS

Theorem 1: Let $r = \max_{i\geq 0}\frac{r_{i+1}}{r_i}$. The number of entries in the n^{th} row of the Fibonacci triangle not divisible by p is $2^{s_1}3^{s_2}4^{s_3}\dots r^{s_{r-1}}$, where s_i is the number of i's in the base $\wp$ expansion of n.

Proof: First, we note that the maximum exists. It is well known that $r_1 \leq p+1$. By (4), we know that $\frac{r_{i+1}}{r_i} \leq p$ for $i \geq 1$, so $r \leq p+1$.

By Kummer's Theorem for Generalized Binomial Coefficients, $p \nmid \left[{n \atop k}\right]_{\mathfrak{F}}$ if and only if there is no carry when k and $n-k$ are added in base $\wp$. Let the base $\wp$ expansions of n and k be $n = (n_t \dots n_2 n_1 n_0)_\wp$ and $k = (k_t \dots k_2 k_1 k_0)_\wp$. Then there is no carry when adding k and $n-k$ in base $\wp$ if and only if $k_i \leq n_i$ for all i. For a fixed n, the number of such k is $\prod_i (n_i+1)$ since there are (n_i+1) possible values of k_i less than or equal to n_i. □

The iterative structure of Pascal's triangle modulo 2 has been studied extensively (see [13]). Recently, the iterative structure of the Fibonacci triangle modulo 2 has also been studied. In particular, a map between the Fibonacci triangle modulo 2 and Pascal's triangle modulo 2 was found in [11]. For all primes $p \neq 2, 5$ whose rank of apparition is even, we get an analogous result: a map between the Fibonacci triangle modulo p and Pascal's triangle modulo p. While the result for these primes is similar to the case $p=2$, our method of proof is different and, in fact, breaks down for $p=2$.

Theorem 2: Let $p \neq 2, 5$ be a prime. There is the following connection between the Fibonacci and binomial coefficients modulo p:

$$\begin{bmatrix} nr_1 \\ kr_1 \end{bmatrix}_{\mathfrak{F}} \equiv \binom{n}{k} F_{r_1+1}^{k(n-k)r_1} \pmod{p}.$$

In particular, the triangle Δ_{F_p} formed by having $\left[{nr_1 \atop kr_1}\right]_{\mathfrak{F}} \pmod{p}$ as the k^{th} entry of the n^{th} row is Pascal's triangle modulo p if and only if r_1 is even.

Proof: By definition

$$\begin{bmatrix} nr_1 \\ kr_1 \end{bmatrix}_{\mathfrak{F}} \equiv \frac{F_{nr_1}F_{nr_1-1}\cdots F_{(n-k)r_1+1}}{F_{kr_1}F_{kr_1-1}\cdots F_2F_1}.$$

Separating the factors divisible by p from those not divisible by p, we get

$$\begin{bmatrix} nr_1 \\ kr_1 \end{bmatrix}_{\mathfrak{F}} \equiv \frac{F_{nr_1}F_{(n-1)r_1}\cdots F_{(n-k+1)r_1}}{F_{kr_1}F_{(k-1)r_1}\cdots F_{r_1}} \cdot \frac{F_{nr_1-1}F_{nr_1-2}\cdots F_{(n-k)r_1+1}}{F_{kr_1-1}F_{kr_1-2}\cdots F_1}.$$

Using Lemmas 3 and 4 to simplify, we obtain

$$\begin{aligned}\begin{bmatrix} nr_1 \\ kr_1 \end{bmatrix}_{\mathfrak{F}} &\equiv \frac{F_{nr_1}F_{(n-1)r_1}\cdots F_{(n-k+1)r_1}}{F_{kr_1}F_{(k-1)r_1}\cdots F_{r_1}} \cdot \frac{F_{nr_1+1}}{F_{kr_1+1}} \cdot \frac{F_{nr_1+1}}{F_{kr_1+1}} \cdot \cdots \cdot \frac{F_{nr_1+1}}{F_{kr_1+1}} \\ &\equiv \frac{F_{nr_1}F_{(n-1)r_1}\cdots F_{(n-k+1)r_1}}{F_{kr_1}F_{(k-1)r_1}\cdots F_{r_1}} \left(\frac{F_{nr_1+1}}{F_{kr_1+1}}\right)^{k(r_1-1)} \pmod{p}.\end{aligned}$$

Using Lemma 4 to simplify further, we get

$$\begin{bmatrix} nr_1 \\ kr_1 \end{bmatrix}_{\mathfrak{F}} \equiv \frac{F_{nr_1}F_{(n-1)r_1}\cdots F_{(n-k+1)r_1}}{F_{kr_1}F_{(k-1)r_1}\cdots F_{r_1}}(F_{(n-k)r_1+1})^{(r_1-1)k} \pmod{p}. \tag{6}$$

Now there are two cases to consider. If the number of factors of p in the numerator of the fraction

$$\frac{F_{nr_1}\cdots F_{(n-k+1)r_1}}{F_{kr_1}\cdots F_{r_1}}$$

is greater than the number of factors of p in the denominator, then $[{nr_1 \atop kr_1}]_{\mathfrak{F}} \equiv 0 \pmod{p}$. But by Kummer's theorem applied to $\mathcal{A} = \{F_{r_1 j}\}_{j=1}^{\infty}$, $p \mid [{n \atop k}]_{\mathcal{A}}$ if and only if there is a carry when adding k and $n-k$ in base $\wp' = \{\rho_0, \rho_1, \rho_2 \ldots\}$, where ρ_i is defined by $p^i \mid F_{r_1 j}$ if and only if $\rho_i \mid j$. By (4), all the ρ_i are powers of p, so there is a carry when adding k and $n-k$ in base $\wp'$ if and only if there is a carry when adding k and $n-k$ in base p (i.e., $\{1, p, p^2, \ldots\}$). By Kummer's Theorem for Generalized Binomial Coefficients, there is a carry when adding k and $n-k$ in base p if and only if $p \mid \binom{n}{k}$. In short, modulo p, the zeros of $[{nr_1 \atop kr_1}]_{\mathfrak{F}}$ correspond to the zeros of $\binom{n}{k}$, since the base $\wp'$ for $\mathcal{A} = \{F_{r_1 j}\}_{j=1}^{\infty}$ is the same, up to repeated terms, as the base corresponding to $\binom{n}{k}$, namely, $\{1, p, p^2, \ldots\}$.

Now consider the case where the number of factors of p in the numerator of the above fraction is the same as the number of factors of p in the denominator. We know that $F_{nr_1} \equiv F_{r_1}(nF_{r_1+1}^{n-1})$ (mod p^2) by Lemma 2, so

$$\frac{F_{nr_1}F_{(n-1)r_1}\cdots F_{(n-k+1)r_1}}{F_{kr_1}F_{(k-1)r_1}\cdots F_{r_1}} \equiv \binom{n}{k}\frac{F_{r_1+1}^{n-1}F_{r_1+1}^{n-2}\cdots F_{r_1+1}^{n-k}}{F_{r_1+1}^{k-1}F_{r_1+1}^{k-2}\cdots F_{r_1+1}^{0}} \equiv \binom{n}{k}F_{r_1+1}^{k(n-k)} \pmod{p}.$$

This means that (6) can be simplified to

$$\begin{bmatrix} nr_1 \\ kr_1 \end{bmatrix}_{\mathfrak{F}} \equiv \binom{n}{k}F_{r_1+1}^{k(n-k)}F_{(n-k)r_1+1}^{(r_1-1)k} \pmod{p}. \tag{7}$$

By Lemma 1, this simplifies to

$$\begin{bmatrix} nr_1 \\ kr_1 \end{bmatrix}_{\mathfrak{F}} \equiv \binom{n}{k}F_{r_1+1}^{k(n-k)}F_{r_1+1}^{(n-k)(r_1-1)k} \equiv \binom{n}{k}F_{r_1+1}^{k(n-k)r_1} \pmod{p}. \tag{8}$$

This proves the first assertion of the theorem.

Now suppose that r_1 is even. By Lemma 5, $F_{r_1+1} \equiv \pm 1 \pmod{p}$. Then (8) reduces to

$$\begin{bmatrix} nr_1 \\ kr_1 \end{bmatrix}_{\mathfrak{F}} \equiv \binom{n}{k} \pmod{p}.$$

Finally, we need to show that when r_1 is odd, Δ_{F_p} is not the same as Pascal's triangle modulo p ($p = 2$ being the lone exception). For this, it is enough to show a single entry that does not match. By (8),

$$\begin{bmatrix} 2r_1 \\ r_1 \end{bmatrix}_{\mathfrak{F}} \equiv \binom{2}{1}F_{r_1+1}^{(1)(2-1)r_1} \pmod{p}.$$

By Lemma 5, when r_1 is odd, F_{r_1+1} has order 4 modulo p. In particular, an odd power of F_{r_1+1} cannot be congruent to 1 modulo p, so

$$\begin{bmatrix} 2r_1 \\ r_1 \end{bmatrix}_{\mathfrak{F}} \not\equiv \binom{2}{1} \pmod{p}. \quad \square$$

In the case $p=5$, we find Δ_{F_5} is the same as the Fibonacci triangle modulo 5. The case $p=2$ was dealt with previously in [11].

We note that there are infinitely many primes p for which Δ_{F_p} is the same as Pascal's triangle modulo p and there are infinitely many for which Δ_{F_p} is not the same as Pascal's triangle modulo p. By Theorem 2, this is equivalent to saying there are infinitely many primes p for which r_1 is even, since there are infinitely many F_{2^i} and for $i \ge 2$ there is always a prime factor of F_{2^i} which is not a prime factor of F_{2^j} for any $j < i$ [this follows from $r_1(2)=3$, $F_{2^i} > F_{2^j}$ for $i > j$ and (4)]. Similarly, F_{3^i}, $i \ge 2$ may be used to show that infinitely many primes p have odd r_1.

As a result of Theorem 2 and Lemma 6, we have the following connection between an arbitrary nonzero Fibonacci coefficient modulo p and a well-defined Fibonacci coefficient in the first r_1 rows of the Fibonacci triangle.

Theorem 3: For $p \ne 2, 5$ a prime, we have

$$\begin{bmatrix} nr_1+j \\ mr_1+i \end{bmatrix}_{\mathfrak{F}} \equiv \binom{n}{m} \begin{bmatrix} j \\ i \end{bmatrix}_{\mathfrak{F}} F_{r_1+1}^{r_1m(n-m)+i(n-m)+m(j-i)} \pmod{p}. \tag{9}$$

Proof: By Lemma 6,

$$\begin{bmatrix} nr_1+j \\ mr_1+i \end{bmatrix}_{\mathfrak{F}} \equiv \begin{bmatrix} nr_1 \\ mr_1 \end{bmatrix}_{\mathfrak{F}} \begin{bmatrix} j \\ i \end{bmatrix}_{\mathfrak{F}} F_{(n-m)r_1-1}^{i} F_{mr_1-1}^{j-i} \pmod{p}.$$

By (8), this becomes

$$\begin{bmatrix} nr_1+j \\ mr_1+i \end{bmatrix}_{\mathfrak{F}} \equiv \binom{n}{m} \begin{bmatrix} j \\ i \end{bmatrix}_{\mathfrak{F}} F_{r_1+1}^{r_1m(n-m)} F_{(n-m)r_1-1}^{i} F_{mr_1-1}^{j-i} \pmod{p}.$$

Applying Lemma 1, we get

$$\begin{aligned} \begin{bmatrix} nr_1+j \\ mr_1+i \end{bmatrix}_{\mathfrak{F}} &\equiv \binom{n}{m} \begin{bmatrix} j \\ i \end{bmatrix}_{\mathfrak{F}} F_{r_1+1}^{r_1m(n-m)} F_{r_1+1}^{i(n-m)} F_{r_1+1}^{m(j-i)} \\ &\equiv \binom{n}{m} \begin{bmatrix} j \\ i \end{bmatrix}_{\mathfrak{F}} F_{r_1+1}^{r_1m(n-m)+i(n-m)+m(j-i)} \pmod{p}. \quad \square \end{aligned}$$

Theorem 3 allows rapid computation of $[{n \atop k}]_{\mathfrak{F}} \pmod{p}$ for large n, k as shown in Examples 1 and 2 in the next section. Theorem 3 may be interpreted geometrically as a relation between columns in rows nr_1 to $nr_1+(r_1-1)$ and the first r_1 rows of the Fibonacci triangle modulo p; each entry in the first r_1 rows is multiplied by the constant $\binom{n}{m}F_{r_1+1}^{r_1m(n-m)+i(n-m)+m(j-i)}$ modulo p to get the corresponding entry between rows nr_1 and $nr_1+(r_1-1)$. This is demonstrated in Example 3 of Section 5.

5. EXAMPLES

Example 1: In order to calculate $\left[\begin{smallmatrix}83\\46\end{smallmatrix}\right]_{\mathfrak{F}}$ (mod 13), we first note that for $p=13$, $r_1=7$, and $F_{r_1-1}=F_6\equiv 8 \pmod{13}$. Then by (9) we have

$$\begin{bmatrix}83\\46\end{bmatrix}_{\mathfrak{F}}=\begin{bmatrix}11(7)+6\\6(7)+4\end{bmatrix}_{\mathfrak{F}}\equiv\binom{11}{6}\begin{bmatrix}6\\4\end{bmatrix}_{\mathfrak{F}}F_6^{7(6)(11-6)} \pmod{13}.$$

Remembering that F_6 has order 4 modulo 13 (since r_1 is odd), we have

$$\begin{bmatrix}83\\46\end{bmatrix}_{\mathfrak{F}}\equiv\binom{11}{6}\begin{bmatrix}6\\4\end{bmatrix}_{\mathfrak{F}}8^{(3)(2)(1)}\equiv\binom{11}{6}\begin{bmatrix}6\\4\end{bmatrix}_{\mathfrak{F}}8^2 \pmod{13}.$$

Since $\binom{11}{6}\equiv 7 \pmod{13}$ and $\left[\begin{smallmatrix}6\\4\end{smallmatrix}\right]_{\mathfrak{F}}\equiv 1 \pmod{13}$, we conclude that

$$\begin{bmatrix}83\\46\end{bmatrix}_{\mathfrak{F}}\equiv 7(1)(-1)\equiv 6 \pmod{13}.$$

Example 2: In order to calculate $\left[\begin{smallmatrix}1000\\768\end{smallmatrix}\right]_{\mathfrak{F}}$ (mod 89), we note for $p=89$, $r_1=11$, and $F_{r_1-1}=F_{10}\equiv 55 \pmod{89}$. Then by (9) we have

$$\begin{bmatrix}1000\\768\end{bmatrix}_{\mathfrak{F}}=\begin{bmatrix}90(11)+10\\69(11)+9\end{bmatrix}_{\mathfrak{F}}\equiv\binom{90}{69}\begin{bmatrix}10\\9\end{bmatrix}_{\mathfrak{F}}F_{10}^{(11)(69)(90-69)} \pmod{89}.$$

Since $\binom{90}{69}\equiv 0 \pmod{89}$ (i.e., a carry occurs when adding 21 and 69 base 89), we conclude that

$$\begin{bmatrix}1000\\768\end{bmatrix}_{\mathfrak{F}}\equiv 0 \pmod{89}.$$

Example 3: Theorem 3 can be interpreted geometrically. For $p=3$ we have $r_1=4$ and $F_{r_1-1}=2\equiv -1 \pmod 3$. The first four rows of the Fibonacci triangle taken modulo 3 are:

$$\begin{array}{ccccccc} &&&1&&& \\ &&1&&1&& \\ &1&&1&&1& \\ 1&&2&&2&&1 \end{array}$$

FIGURE 2. Basic Triangle Modulo 3

By Theorem 3, this 4-row triangle with variations based on the parity of m and n will build the entire Fibonacci triangle modulo 3. Specifically for the 4 cases of m, n even or odd, we have

$$\begin{array}{cccc} 1 & 1 & 1 & 1 \\ 1\ 1 & 1\ 2 & 2\ 1 & 2\ 2 \\ 1\ 1\ 1 & 1\ 2\ 1 & 1\ 2\ 1 & 1\ 1\ 1 \\ 1\ 2\ 2\ 1 & 1\ 1\ 2\ 2 & 2\ 2\ 1\ 1 & 2\ 1\ 1\ 2 \\ m, n \text{ even} & m \text{ even} & \text{neither} & n \text{ even} \end{array}$$

FIGURE 3. The Four Variants of the Basic Triangle Modulo 3

For example, the triangle in rows 4 to 7 ($n=1$) and columns 0 to 3 ($m=0$) is the second triangle in Figure 3 with entry multiplied by $\binom{1}{0}=1$. The triangle in rows 8 to 11 ($n=2$) and columns 4 to 7 ($m=1$) is the fourth triangle in Figure 3 with each entry multiplied by $\binom{2}{1}=2$. These are shown in Figure 4.

$$\begin{array}{c}
1 \\
1\ 1 \\
1\ 1\ 1 \\
1\ 2\ 2\ 1 \\
1\ 0\ 0\ 0\ 1 \\
1\ 2\ 0\ 0\ 2\ 1 \\
1\ 2\ 1\ 0\ 1\ 2\ 1 \\
1\ 1\ 2\ 2\ 2\ 2\ 1\ 1 \\
1\ 0\ 0\ 0\ 2\ 0\ 0\ 0\ 1 \\
1\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 1 \\
1\ 1\ 1\ 0\ 2\ 2\ 2\ 0\ 1\ 1\ 1 \\
1\ 2\ 2\ 1\ 1\ 2\ 2\ 1\ 1\ 2\ 2\ 1
\end{array}$$

FIGURE 4. Rows 0 to 11 of the Fibonacci Triangle Modulo 3

More generally, to determine the triangle in rows $4n$ to $4n+3$ and columns $4m$ to $4m+3$, we pick the appropriate triangle in Figure 3, based on the parity of m, n and multiply each entry by $\binom{n}{m}$ (mod 3).

ACKNOWLEDGMENTS

The author would like to thank two of his MA students, Rob and Nicole Fraser, for preliminary discussions in general and for conjecturing Theorem 1 in particular.

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AMS Classification Numbers: 11B39, 11B65